Final exam for Kwantumfysica 1 - 2011-2012

Thursday 3 November 2011, 9:00 - 12:00

<u>READ THIS FIRST</u>:

- Note that the lower half of this page lists some useful formulas and constants.
- Clearly write your name and study number on each answer sheet that you use.
- On the first answer sheet, write the total number of answer sheets that you turn in.
- The exam has several questions, it continues on the backside of the papers.
- Start each question (number 1, 2, etc.) on a new answer sheet.
- The exam is open book with limits. You are allowed to use the book by Griffiths or Liboff, the handouts *Extra note on two-level systems and exchange degeneracy for identical particles*, and *Feynman Lectures chapter III-1*, one A4 sheet with your own notes, but nothing more than this.
- If it says "make a rough estimate", there is no need to make a detailed calculation, and making a simple estimate is good enough. If it says "calculate" or "derive", you are supposed to present a full analytical calculation.
- If you get stuck on some part of a problem for a long time, it may be wise to skip it and try the next part of a problem first.
- If you are ready with the exam, please fill in the **course-evaluation question sheet**. You can keep working on the exam until the scheduled end time, and fill it in shortly after that if you like.

Useful formulas and constants:

Electron mass	m_e	$= 9.1 \cdot 10^{-31}$ kg
Electron charge	$-e$	$= -1.6 \cdot 10^{-19}$ C
Planck's constant	h	$= 6.626 \cdot 10^{-34}$ Js $= 4.136 \cdot 10^{-15}$ eVs
Planck's reduced constant	$\hbar$	$= 1.055 \cdot 10^{-34}$ Js $= 6.582 \cdot 10^{-16}$ eVs

Fourier relation between x-representation and k-representation of a state

$$\Psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{\Psi}(k) e^{ikx}\, dk$$

$$\overline{\Psi}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x) e^{-ikx}\, dx$$

Standard Fourier transform pairs:

$$\Psi(x) = \begin{cases} \frac{1}{\sqrt{2b}}, & |x| \le b \\ 0, & |x| > b \end{cases} \quad \overset{Fourier}{\leftrightarrow} \quad \overline{\Psi}(k) = \sqrt{\frac{b}{\pi}} \frac{\sin kb}{kb}$$

$$\Psi(x) = \sqrt{\frac{b}{\pi}} \frac{\sin bx}{bx} \quad \overset{Fourier}{\leftrightarrow} \quad \overline{\Psi}(k) = \begin{cases} \frac{1}{\sqrt{2b}}, & |k| \le b \\ 0, & |k| > b \end{cases}$$

Standard integrals:

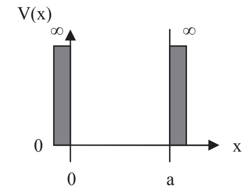
$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

$$\int_{-\infty}^{\infty} x^2 e^{-x^2} dx = \tfrac{1}{2}\sqrt{\pi}$$

Problem 1

Consider a particle with mass m in a one-dimensional square well with the following potential:

$$V(x) = \begin{cases} \infty & for \quad x < 0 \quad or \quad x > a \\ 0 & for \quad 0 < x < a \end{cases}$$



a) Write down the Hamiltonian of this system in x-representation, and write out each term in as much detail as you can.
<u>Also</u> write down the eigenstates and eigenenergies of this system.

b) At a time $t = 0$ a particle is in a superposition of the ground state and first excited state of this system: $|\Psi(t=0)\rangle = \frac{1}{\sqrt{2}}\left(|\varphi_1\rangle + |\varphi_2\rangle\right)$. What is:

b.1) the expectation value for the total energy.
b.2) $|\Psi(t)\rangle$ (you can either use Dirac notation or x-representation, but otherwise work it out in as much detail as you can).
b.3) the expectation value of the energy at a time $t > 0$.

c) From a measurement of the energy of the system at $t = t_0$ we got $E = \dfrac{\pi^2 \hbar^2}{2ma^2}$, what is the state of the system just after the measurement? Make a sketch of the wavefunction of this state.

d) Just after measuring the energy, the right wall of the well moves *very quickly* from the position $x = a$ to $x = 2a$. This change happens so quickly that the wavefunction of the particle *does not have time to change* during the displacement of the wall. Make a sketch of the wavefunction immediately after we moved the wall.

e) Now make another sketch of the ground state and first excited state of the new well with width $2a$.

f) Assume again the situation immediately after we moved the wall. Using just simple arguments, present a conceptual/qualitative discussion that explains whether the probability of finding the particle in the ground state is smaller or larger than the probability of finding the particle in the first excited state. *Hint: look at your two previous sketches and remember what it means to analyze the projection of one state onto another. Also note that you are asked to calculate the exact result in question* g).

g) Calculate the two probabilities of question **f)** and check if your predictions were correct. You might need the following trigonometric relation:

$$\sin(a)\sin(b) = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$$

Problem 2

Consider a hydrogen atom. Assume that the electron is in a state for which the orbital wavefunction has the quantum numbers $n = 4$ and $l = 2$ (using the usual notation). In this problem you need to consider the electron as a particle that has spin.

a) We have an instrument that can be used to measure to the z-component of angular momentum or to measure the length of an angular momentum vector. Before each measurement the electron is first prepared in the state with $n = 4$ and $l = 2$.

a1) The instrument is tuned to measure the z-component of the spin of the electron. What are the possible measurement outcomes?

a2) The instrument is tuned to measure the length of the angular momentum vector while only measuring on the spin of the electron. What are the possible measurement outcomes?

a3) The instrument is tuned to measure the z-component of the orbital angular momentum of the electron. What are the possible measurement outcomes?

a4) The instrument is tuned to measure the length of the angular momentum vector while only measuring on the orbital angular momentum of the electron. What are the possible measurement outcomes?

a5) A similar instrument can be used to measure to the z-component of the magnetic dipole moment of the electron (from the electron on itself, it is not measuring a magnetic dipole moment from the orbital state). What are the possible measurement outcomes if this instrument is used?

b) Now, the same instrument as in questions **a1)-a4)** is tuned to measure the *total amount of angular momentum in the atom* (the addition of orbital and spin contributions). In this problem you are supposed to ignore the angular momentum of the nucleus. Assume again that before each measurement the electron is first prepared in the state with $n = 4$ and $l = 2$. What are now the possible measurement outcomes when:

b1) we measure the length of the angular momentum vector of the atom's total amount of angular momentum.

b2) we measure the z-component of the total amount of angular momentum.

Problem 3

A certain atom is in a state with its total angular momentum vector $\boldsymbol{L}$ (described by the operator $\hat{L}$) defined by quantum number $l = 1$. For the system in this state, the operator for the z-component of angular momentum is $\hat{L}_z$. It has three eigenvalues, $+\hbar$ (with corresponding eigenstate $|+_z\rangle$), $0\hbar$ (with eigenstate $|0_z\rangle$), and $-\hbar$ (with eigenstate $|-_z\rangle$). This operator can be represented as a matrix, and the ket-states as column vectors, using the basis spanned by $|+_z\rangle$, $|0_z\rangle$ and $|-_z\rangle$, according to

$$\hat{L}_z \leftrightarrow \hbar\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}, \quad |+_z\rangle \leftrightarrow \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad |0_z\rangle \leftrightarrow \begin{pmatrix}0\\1\\0\end{pmatrix}, \text{ and } \quad |-_z\rangle \leftrightarrow \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

Using this same basis for the representation, the operator and eigenstates for the system's x-component of angular momentum are given by

$$\hat{L}_x \leftrightarrow \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}, \quad |+_x\rangle \leftrightarrow \begin{pmatrix}\frac{1}{2}\\ \frac{1}{\sqrt{2}}\\ \frac{1}{2}\end{pmatrix}, \quad |0_x\rangle \leftrightarrow \begin{pmatrix}\frac{1}{\sqrt{2}}\\ 0\\ -\frac{1}{\sqrt{2}}\end{pmatrix}, \text{ and } \quad |-_x\rangle \leftrightarrow \begin{pmatrix}\frac{1}{2}\\ -\frac{1}{\sqrt{2}}\\ \frac{1}{2}\end{pmatrix}.$$

a) At some point the system is in the normalized state $|\Psi_1\rangle = \sqrt{\frac{1}{5}}\,|+_z\rangle + \sqrt{\frac{1}{5}}\,|0_z\rangle + i\,\sqrt{\frac{3}{5}}\,|-_z\rangle$. Calculate for this state the expectation value for angular momentum in z-direction and the expectation value for angular momentum in x-direction.

b) With the system still in this same state $|\Psi_1\rangle$, you are going to measure the x-component of the system's angular momentum. What are the possible measurement results? Calculate the probability for getting the measurement result with the highest value for angular momentum in x-direction.

c) Now the system is prepared in a different state (now superposition of eigenstates of $\hat{L}_x$), $|\Psi_2\rangle = \sqrt{\frac{1}{3}}\,|+_x\rangle + \sqrt{\frac{2}{3}}\,|-_x\rangle$. You are going to measure the z-component of the system's angular momentum. What is the probability to find the answer $+\hbar$?

d) Now the system is prepared in a different state (now again a superposition of eigenstates of $\hat{L}_z$), $|\Psi_3\rangle = \sqrt{\frac{1}{2}}\,|+_z\rangle + \sqrt{\frac{1}{2}}\,|0_z\rangle$, at time $t = 0$. Another change to the system is that one now applied an external magnetic field with magnitude B along the z-axis. The Hamiltonian of the system is now, $\hat{H} = \gamma B\hat{L}_z$, where γ is a constant that reflects how much the energy of angular momentum states shifts when applying the field. Calculate how the expectation value for angular momentum in x-direction depends on time. Use Dirac notation and the operator $\hat{U} = e^{\frac{-i\hat{H}t}{\hbar}}$.
Also describe and discuss (in words, use about 4 lines of text) the dynamics of the angular momentum vector $\boldsymbol{L}$ that occurs in this situation.

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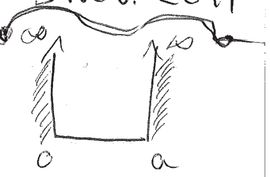
Problem 1

$$V(x) = \begin{cases} \infty & \text{for } x<0 \text{ or } x>a \\ 0 & \text{for } 0<x<a \end{cases}$$



a) $$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$$

with $$V(x) = \begin{cases} \infty & \text{for } x<0 \text{ or } x>a \\ 0 & \text{for } 0<x<a \end{cases}$$

The eigenstates are $|\varphi_n\rangle \leftrightarrow \varphi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$

with eigenenergies $\epsilon_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$: $\hat{H}|\varphi_n\rangle = \epsilon_n|\varphi_n\rangle$

b) $$|\Psi(t=0)\rangle = \frac{1}{\sqrt{2}}\left[|\varphi_1\rangle + |\varphi_2\rangle\right]$$

b.1) $$\langle\hat{H}\rangle = \langle\Psi(t=0)|\hat{H}|\Psi(t=0)\rangle =$$

$$= \left[\frac{1}{\sqrt{2}}\left(\langle\varphi_1| + \langle\varphi_2|\right)\right]\hat{H}\left(\frac{1}{\sqrt{2}}\left(|\varphi_1\rangle + |\varphi_2\rangle\right)\right) =$$

$$= \frac{1}{2}\left[\langle\varphi_1|\hat{H}|\varphi_1\rangle + \langle\varphi_1|\hat{H}|\varphi_2\rangle + \langle\varphi_2|\hat{H}|\varphi_1\rangle + \langle\varphi_2|\hat{H}|\varphi_2\rangle\right]$$

Since $\hat{H}|\varphi_n\rangle = \epsilon_n|\varphi_n\rangle \Rightarrow \langle\varphi_n|\hat{H}|\varphi_n\rangle = \epsilon_n$
and $\langle\varphi_n|\varphi_m\rangle = \delta_{mn}$, we have
$\langle\varphi_n|\hat{H}|\varphi_m\rangle = \epsilon_m\delta_{mn}$

So:
$$\langle\hat{H}\rangle = \frac{1}{2}\left[\epsilon_{n=1} + 0 + 0 + \epsilon_{n=2}\right]$$
$$= \frac{1}{2}\left[\frac{1\pi^2\hbar^2}{2ma^2} + \frac{2^2\pi^2\hbar^2}{2ma^2}\right]$$

$$\boxed{\langle\hat{H}\rangle = \frac{5\pi^2\hbar^2}{4ma^2}} \text{ or } \boxed{\langle\hat{H}\rangle = \frac{\epsilon_1+\epsilon_2}{2} = \frac{5}{2}\epsilon_1}$$

b.2) $$|\Psi(t)\rangle = e^{-\frac{i\hat{H}t}{\hbar}}|\Psi(t=0)\rangle$$

$$e^{-\frac{i\hat{H}t}{\hbar}}|\varphi_n\rangle = e^{-\frac{i\epsilon_n t}{\hbar}}|\varphi_n\rangle, \text{ so:}$$

$$|\Psi(t)\rangle = e^{-\frac{i\hat{H}t}{\hbar}}\left[\frac{1}{\sqrt{2}}\left(|\varphi_1\rangle + |\varphi_2\rangle\right)\right]$$

$$= \frac{1}{\sqrt{2}}\left[e^{-\frac{i\hat{H}t}{\hbar}}|\varphi_1\rangle + e^{-\frac{i\hat{H}t}{\hbar}}|\varphi_2\rangle\right]$$

$$\boxed{|\Psi(t)\rangle = \frac{1}{\sqrt{2}}\left[e^{-\frac{i\epsilon_1 t}{\hbar}}|\varphi_1\rangle + e^{-\frac{i\epsilon_2 t}{\hbar}}|\varphi_2\rangle\right]}$$

b.3) $$\langle\hat{H}(t)\rangle = \langle\Psi(t)|\hat{H}|\Psi(t)\rangle =$$

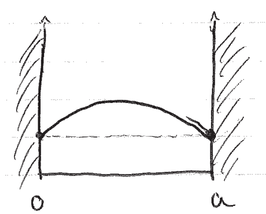
$$= \frac{1}{2}\left[e^{+\frac{i\epsilon_1 t}{\hbar}}\langle\varphi_1| + e^{+\frac{i\epsilon_2 t}{\hbar}}\langle\varphi_2|\right]\hat{H}\left[e^{-\frac{i\epsilon_1 t}{\hbar}}|\varphi_1\rangle + e^{-\frac{i\epsilon_2 t}{\hbar}}|\varphi_2\rangle\right]$$

$$= \frac{1}{2}\left[1\langle\varphi_1|\hat{H}|\varphi_1\rangle + e^{\frac{i(\epsilon_1-\epsilon_2)t}{\hbar}}\cancelto{0}{\langle\varphi_1|\hat{H}|\varphi_2\rangle} + e^{-\frac{i(\epsilon_1-\epsilon_2)t}{\hbar}}\cancelto{0}{\langle\varphi_2|\hat{H}|\varphi_1\rangle} + 1\langle\varphi_2|\hat{H}|\varphi_2\rangle\right]$$

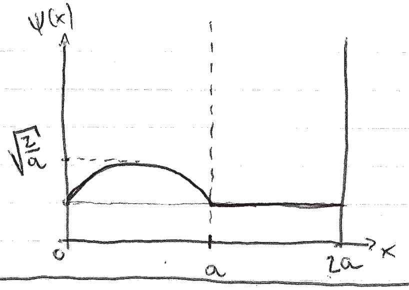
$$= \frac{1}{2}\left[\epsilon_1 + \epsilon_2\right] \Rightarrow \boxed{\langle\hat{H}(t)\rangle = \frac{\epsilon_1+\epsilon_2}{2}}$$

c) $\epsilon = \epsilon_1$ So the particle is at $|\varphi_1\rangle$

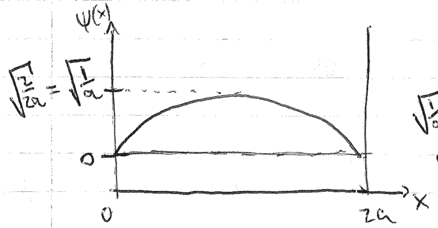
$$|\Psi\rangle = |\varphi_1\rangle \leftrightarrow \sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right) = \varphi_1(x)$$



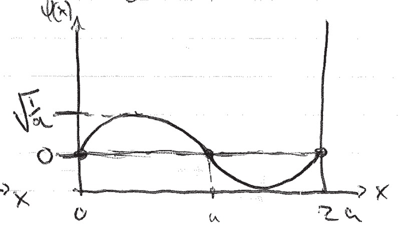
d)



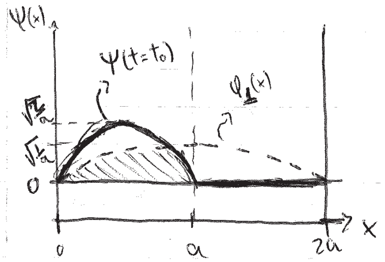
e) Ground state

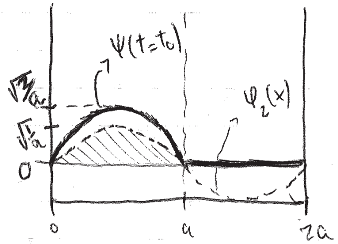


1^{st} excited state



f) The projection of one state into another: $c_n = \langle \Psi | \phi_n \rangle$ represents "how much" of the state $|\phi_n\rangle$ is contained in $|\Psi\rangle$. And the probability of getting $|\phi_n\rangle$ when measuring $|\Psi\rangle$ is $|c_n|^2$. Now let's check the sketches:





The coeficients c_n are proportional to the shaded areas in the graphs above. From this we can see that $c_2 > c_1$. Since the total area under $|\varphi_1(x)|^2 = |\varphi_2(x)|^2 = 1$ we have that $|c_2|^2 = 0.5$ and $|c_1|^2 < 0.5$.

g) $$c_1 = \langle \Psi | \varphi_1 \rangle = \int_0^a \left[\sqrt{\tfrac{2}{a}} \sin\left(\tfrac{\pi x}{a}\right)\right] \cdot \left[\sqrt{\tfrac{1}{a}} \sin\left(\tfrac{\pi x}{2a}\right)\right] dx + \int_a^{2a} \emptyset \, dx$$

$$= \frac{\sqrt{2}}{a} \int_0^a \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi x}{2a}\right) dx =$$

$$= \frac{\sqrt{2}}{a} \frac{1}{2} \left[\int_0^a \cos\left(\frac{\pi x}{2a}\right) dx - \int_0^a \cos\left(\frac{3\pi x}{2a}\right) dx \right] =$$

$$= \frac{\sqrt{2}}{2a} \left[\frac{2a}{\pi} \sin\left(\frac{\pi x}{2a}\right)\Big|_0^a - \frac{2a}{3\pi} \sin\left(\frac{3\pi x}{2a}\right)\Big|_0^a \right] =$$

$$= \frac{\sqrt{2}}{\cancel{2a}} \left[\frac{\cancel{2a}}{\pi}(1-0) - \frac{\cancel{2a}}{3\pi}(-1-0) \right] = \sqrt{2}\left[\frac{3+1}{3\pi}\right] = \frac{4\sqrt{2}}{3\pi}$$

$$\boxed{P_{n=1} = |c_1|^2 = \frac{32}{9\pi^2} \approx 0.36}$$

$$c_2 = \langle \Psi | \varphi_2 \rangle = \int_0^a \sqrt{\tfrac{2}{a}} \sin\left(\tfrac{\pi x}{a}\right) \sqrt{\tfrac{1}{a}} \sin\left(\tfrac{2\pi x}{2a}\right) dx + \int_a^{2a} \emptyset \, dx =$$

$$= \frac{\sqrt{2}}{a} \int_0^a \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi x}{a}\right) dx =$$

$$= \frac{\sqrt{2}}{a} \frac{1}{2} \left[\int_0^a \cos(0)\, dx - \int_0^a \cos\left(\frac{2\pi x}{a}\right) dx \right] =$$

$$= \frac{\sqrt{2}}{2a} \left[(a-0) - \frac{a}{2\pi} \sin\left(\frac{2\pi x}{a}\right)\Big|_0^a \right] = \frac{\sqrt{2}}{2a}\left[a - 0\right]$$

$$c_2 = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} \Rightarrow \boxed{P_{n=2} = |c_2|^2 = \frac{1}{2} = 0.5}$$

$c_1 < c_2$ and $|c_2|^2 = 0.5$, $|c_1|^2 < 0.5$ ✓

Problem 2

a) Possible measurement outcomes are determined by the set of eigen values of the associated eigen value equations

For the electron spin

length $\hat{S}^2 |s\, m_s\rangle = \hbar^2 s(s+1) |s\, m_s\rangle$ [Eq 1]

with $s=\frac{1}{2}$, and for any m_s

z-comp. $\hat{S}_z |s\, m_s\rangle = \hbar\, m_s |s\, m_s\rangle$ [Eq 2]

with $s=\frac{1}{2}$ and $m_s = \pm\frac{1}{2}$

For the orbital angular momentum

length $\hat{L}^2 |l\, m_l\rangle = \hbar^2 l(l+1) |l\, m_l\rangle$ [Eq. 3]

with $l=2$ and for any m_l

z-comp $\hat{L}_z |l\, m_l\rangle = \hbar\, m_l |l\, m_l\rangle$ [Eq. 4]

with $l=2$ and for $m_l = -2, -1, 0, +1, +2$

a1) Using Eq 2 this gives as possible outcomes $S_z = \pm\frac{1}{2}\hbar$

a2) Using Eq 1, this gives only one outcome

$$|\vec{S}| = \hbar\sqrt{\tfrac{1}{2}(1+\tfrac{1}{2})} = \sqrt{\tfrac{3}{4}}\,\hbar$$

a3) Using Eq 4 possible outcomes are

$L_z = -2\hbar,\ -1\hbar,\ 0\hbar,\ +1\hbar,\ +2\hbar$

a4) Using Eq 3, the only possible outcome is

$$|\vec{L}| = \hbar\sqrt{l(l+1)} = \sqrt{6}\,\hbar$$

a5) The magnetic dipole moment $\vec{\mu}$ is proportional to the spin

$\hat{\mu} = \gamma\hat{S}$ and $\hat{\mu}_z = \gamma\hat{S}_z$

So, with a1) it follows that the possible outcomes are

$$\mu_z = \pm\tfrac{1}{2}\gamma\hbar$$

b) Use the rules for addition of angular momentum.

with $\vec{J} = \vec{L} + \vec{S}$, and $\vec{J}$ the total angular momentum of the atom.

$\vec{J}$ obeys the eigenvalue equations

$$\boxed{\begin{aligned}\hat{J}^2|j\, m_j\rangle &= \hbar^2 j(j+1)|j\, m_j\rangle \\ \hat{J}_z|j\, m_j\rangle &= \hbar\, m_j\, |j\, m_j\rangle\end{aligned}}\qquad\begin{aligned}&[\text{Eq } 5]\\&[\text{Eq } 6]\end{aligned}$$

with $j = |l+s|,\ |l+s|-1 \ \dots\ |l-s|+1,\ |l-s|$

$\Rightarrow\ j = 2\tfrac{1}{2},\ 1\tfrac{1}{2}$

and $m_j = -j,\ -(j-1)\ \dots\ (j-1),\ j$

b₁) Using Eq 5 gives as possible outcomes

$$|\vec{J}| = \hbar\sqrt{\tfrac{5}{2}\cdot\tfrac{7}{2}} = \sqrt{\tfrac{35}{4}}\,\hbar \text{ and } \hbar\sqrt{\tfrac{3}{2}\cdot\tfrac{5}{2}} = \sqrt{\tfrac{15}{4}}\,\hbar$$

b₂) Listing the outcomes for $j = \frac{5}{2}$ and $\frac{3}{2}$ together, with Eq. 6 gives as possible outcomes

$$J_z = -\tfrac{5}{2}\hbar,\ -\tfrac{3}{2}\hbar,\ -\tfrac{1}{2}\hbar,\ +\tfrac{1}{2}\hbar,\ +\tfrac{3}{2}\hbar,\ +\tfrac{5}{2}\hbar$$

Problem 3

a) $\langle \hat{L}_z \rangle = \langle \psi_1 | \hat{L}_z | \psi_1 \rangle$

$$= \left(\sqrt{\tfrac{1}{5}}\ \ \sqrt{\tfrac{1}{5}}\ \ -i\sqrt{\tfrac{3}{5}}\right)\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}\begin{pmatrix}\sqrt{\tfrac{1}{5}}\\ \sqrt{\tfrac{1}{5}}\\ +i\sqrt{\tfrac{3}{5}}\end{pmatrix}\hbar = \left(\sqrt{\tfrac{1}{5}}\ \ \sqrt{\tfrac{1}{5}}\ \ -i\sqrt{\tfrac{3}{5}}\right)\begin{pmatrix}\sqrt{\tfrac{1}{5}}\\ 0\\ -i\sqrt{\tfrac{3}{5}}\end{pmatrix}\hbar$$

$$= \tfrac{1}{5}\hbar - \tfrac{3}{5}\hbar = -\tfrac{2}{5}\hbar$$

$\langle \hat{L}_x \rangle = \langle \psi_1 | \hat{L}_x | \psi_1 \rangle$

$$= \left(\sqrt{\tfrac{1}{5}}\ \ \sqrt{\tfrac{1}{5}}\ \ -i\sqrt{\tfrac{3}{5}}\right)\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\frac{\hbar}{\sqrt{2}}\begin{pmatrix}\sqrt{\tfrac{1}{5}}\\ \sqrt{\tfrac{1}{5}}\\ +i\sqrt{\tfrac{3}{5}}\end{pmatrix} = \left(\sqrt{\tfrac{1}{5}}\ \ \sqrt{\tfrac{1}{5}}\ \ -i\sqrt{\tfrac{3}{5}}\right)\begin{pmatrix}\sqrt{\tfrac{1}{5}}\\ \sqrt{\tfrac{1}{5}}+i\sqrt{\tfrac{3}{5}}\\ \sqrt{\tfrac{1}{5}}\end{pmatrix}\frac{\hbar}{\sqrt{2}}$$

$$= +\frac{\sqrt{2}}{5}\hbar$$

b) As for $\hat{L}_z$, the set of eigenvalues for $\hat{L}_x = -\hbar, 0, +\hbar$, and you can directly check that eigen value $+\hbar$ belongs to the eigenvector $|+_x\rangle$.

highest eigen value

$$P_{+\hbar} = |\langle +_x | \psi_1 \rangle|^2 = \left| \left(\tfrac{1}{2}\ \ \tfrac{1}{\sqrt{2}}\ \ \tfrac{1}{2}\right)\begin{pmatrix}\sqrt{\tfrac{1}{5}}\\ \sqrt{\tfrac{1}{5}}\\ +i\sqrt{\tfrac{3}{5}}\end{pmatrix} \right|^2$$

$$= \left| \tfrac{1}{2}\sqrt{\tfrac{1}{5}} + \sqrt{\tfrac{1}{10}} + i\tfrac{1}{2}\sqrt{\tfrac{3}{5}} \right|^2 = \frac{3+\sqrt{2}}{10} \simeq 0.44$$

c)

$$P_{+h} = \left| \langle +_z | \psi_2 \rangle \right|^2$$

$$= \left| (1\,0\,0) \left(\sqrt{\tfrac{1}{3}} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} + \sqrt{\tfrac{2}{3}} \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} \right) \right|^2$$

$$= \left| \sqrt{\tfrac{1}{3}}\,\tfrac{1}{2} + \sqrt{\tfrac{2}{3}}\,\tfrac{1}{2} \right|^2 = \frac{\left(1+\sqrt{2}\right)^2}{12} \approx 0.49$$

d) $\langle L_x(t) \rangle = \langle \psi(t) | \hat{L}_x | \psi(t) \rangle$ with $|\psi(t)\rangle = \hat{U} |\psi_3\rangle$

$$\hat{H} \leftrightarrow \begin{pmatrix} E_+ & 0 & 0 \\ 0 & E_0 & 0 \\ 0 & 0 & E_- \end{pmatrix} = \hbar \begin{pmatrix} \omega_+ & 0 & 0 \\ 0 & \omega_0 & 0 \\ 0 & 0 & \omega_- \end{pmatrix} \text{ with } \begin{cases} E_+ = +\gamma B \hbar \\ E_0 = 0 \\ E_- = -\gamma B \hbar \\ \omega_+ = +\gamma B \\ \omega_0 = 0 \\ \omega_- = -\gamma B \end{cases}$$

$$\Rightarrow |\psi(t)\rangle \frac{e^{i\omega_+ t}}{\sqrt{2}} |+_z\rangle + e^{\frac{-i\omega_0 t}{\sqrt{2}}} |0_z\rangle$$

$$\Rightarrow \langle L_x(t) \rangle = \left(e^{+i\omega_+ t}, e^{+i\omega_0 t}, 0 \right) \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} e^{-i\omega_+ t} \\ e^{-i\omega_0 t} \\ 0 \end{pmatrix} \frac{\hbar}{\sqrt{2}} \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}}$$

$$= \frac{\hbar}{2\sqrt{2}} \left(e^{+i(\omega_+ - \omega_0)t} + e^{-i(\omega_+ - \omega_0)t} \right)$$

$$= \frac{\hbar}{\sqrt{2}} \cos\left((\omega_+ - \omega_0) t \right) = \frac{\hbar}{\sqrt{2}} \cos(\gamma B t)$$

The vector $\vec{L}$ precess about the z-axis in the x-y plane ($\langle \hat{L}_y(t) \rangle$ also oscillates like $\langle \hat{L}_x \rangle$, but $\pi/2$ out of phase).
Here we only calculate the full oscillation of $\langle L_x \rangle$ around $L_x = 0$ in a harmonic way.