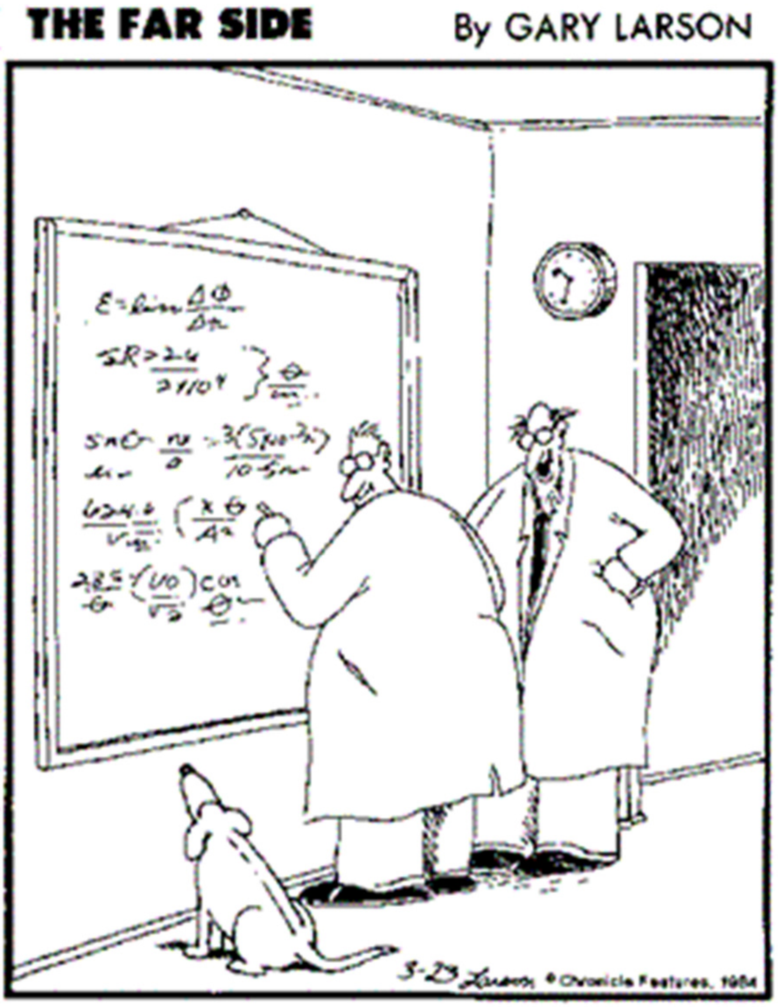
Answer set for the course based on the Griffiths book 2nd Ed.
Last update done on 25 Sept. 2014 - prof. Caspar van der Wal

Quantum Physics 1

Answers to the problem set for the 4th week of the course



"Ohhhhhhh . . . Look at that, Schuster . . . Dogs are so cute when they try to comprehend quantum mechanics."

ANSWERS Problem set WEEK4 Quantum Physics 1

1

Problem W4.1

a) E_g and E_e should be consistent with $|\varphi_g\rangle$ and $|\varphi_e\rangle$ in the Schrödinger equation $\hat{H}|\varphi_i\rangle = E_i|\varphi_i\rangle$

For $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ this gives

$$\begin{pmatrix} E_0 & T \\ T & E_0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = E_+\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \Rightarrow E_+ = E_0 + T$$

For $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$ this gives

$$\begin{pmatrix} E_0 & T \\ T & E_0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} = E_-\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} \Rightarrow E_- = E_0 - T$$

Given that T real and $T<0$, it must be that

$$\begin{cases} E_g = E_+ = E_0 + T, & \text{for } |\varphi_g\rangle \\ E_e = E_- = E_0 - T, & \text{for } |\varphi_e\rangle \end{cases}$$

b) $\langle\varphi_g|\varphi_g\rangle = \left(\frac{1}{\sqrt{2}} \;\; \frac{1}{\sqrt{2}}\right)\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{2}+\frac{1}{2} = 1 \Rightarrow$ Normalized

$\langle\varphi_e|\varphi_e\rangle = \left(\frac{1}{\sqrt{2}} \;\; \frac{-1}{\sqrt{2}}\right)\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{2}+\frac{1}{2} = 1 \Rightarrow$ Normalized

$\langle\varphi_e|\varphi_g\rangle = \left(\frac{1}{\sqrt{2}} \;\; \frac{-1}{\sqrt{2}}\right)\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{2}-\frac{1}{2} = 0 \Rightarrow$ Orthogonal

and $\langle\varphi_g|\varphi_e\rangle = \langle\varphi_e|\varphi_g\rangle^* = 0$

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c) For $\hat{H}_0$:

$$[\hat{A}, \hat{H}_0] = \hat{A}\hat{H}_0 - \hat{H}_0\hat{A} = \begin{pmatrix} -a & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} E_0 & 0 \\ 0 & E_0 \end{pmatrix} - \begin{pmatrix} E_0 & 0 \\ 0 & E_0 \end{pmatrix}\begin{pmatrix} -a & 0 \\ 0 & a \end{pmatrix}$$

$$= \begin{pmatrix} -aE_0 & 0 \\ 0 & aE_0 \end{pmatrix} - \begin{pmatrix} -aE_0 & 0 \\ 0 & aE_0 \end{pmatrix} = 0 \quad \Rightarrow \hat{A} \text{ and } H_0 \text{ commute}$$

For $\hat{H}$:

$$[\hat{A}, \hat{H}] = \hat{A}\hat{H} - \hat{H}\hat{A} = \begin{pmatrix} -a & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} E_0 & T \\ T & E_0 \end{pmatrix} - \begin{pmatrix} E_0 & T \\ T & E_0 \end{pmatrix}\begin{pmatrix} -a & 0 \\ 0 & a \end{pmatrix}$$

$$= \begin{pmatrix} -aE_0 & -aT \\ aT & aE_0 \end{pmatrix} - \begin{pmatrix} -aE_0 & aT \\ -aT & aE_0 \end{pmatrix} = \begin{pmatrix} 0 & -2aT \\ -2aT & 0 \end{pmatrix} \neq 0$$

$\Rightarrow \hat{A}$ and $\hat{H}$ do not commute.

d) $\hat{A}$ is a diagonal matrix, so the eigenvalues are on the diagonal.
$\hat{H}_0$ and $\hat{A}$ commute (but $\hat{H}_0$ degenerate), so the eigenvectors of $\hat{A}$ are the same or a linear superposition of those of $\hat{H}_0$.

$$\hat{A}|\varphi_i\rangle = \pm a |\varphi_i\rangle \quad \Rightarrow$$

$\begin{pmatrix} -a & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = +a\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is consistent for $|\varphi_i\rangle \leftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\Rightarrow$ eigenvalue $+a$ has eigenvector $|\varphi_R\rangle$

$\begin{pmatrix} -a & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -a\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is consistent for $|\varphi_i\rangle \leftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\Rightarrow$ eigenvalue $-a$ has eigenvector $|\varphi_L\rangle$

e) Ground state of $\hat{H}$ is $|\varphi_g\rangle = \frac{1}{\sqrt{2}}\left(|\varphi_L\rangle + |\varphi_R\rangle\right)$

So, a measurement of $\hat{A}$ can give both $+a$ and $-a$ as answer

measurement outcome	Probability	state after measurement
$-a$	$\lvert\langle\varphi_L\lvert\varphi_g\rangle\rvert^2 = \frac{1}{2}$	$\lvert\varphi_L\rangle$
$+a$	$\lvert\langle\varphi_R\lvert\varphi_g\rangle\rvert^2 = \frac{1}{2}$	$\lvert\varphi_R\rangle$
	$\overline{}\;+$ 1	

f) $|\psi\rangle = \sqrt{\frac{1}{3}}\,|\varphi_g\rangle + \sqrt{\frac{2}{3}}\,|\varphi_e\rangle = \left(\sqrt{\frac{1}{6}}\,|\varphi_L\rangle + \sqrt{\frac{1}{6}}\,|\varphi_R\rangle\right) + \left(\sqrt{\frac{2}{6}}\,|\varphi_L\rangle - \sqrt{\frac{2}{6}}\,|\varphi_R\rangle\right)$

$= \frac{1+\sqrt{2}}{\sqrt{6}}\,|\varphi_L\rangle + \frac{1-\sqrt{2}}{\sqrt{6}}\,|\varphi_R\rangle \Rightarrow$ Both $|\varphi_L\rangle$ and $|\varphi_R\rangle$

have non-zero probability amplitude, so a measurement can give both $+a$ and $-a$ as answer.

Probability for $-a$ is $|\langle\varphi_L|\psi\rangle|^2$,
for $+a$ is $|\langle\varphi_R|\psi\rangle|^2$

measurement outcome	Probability	State after measurement
$-a$	$\left(\frac{1+\sqrt{2}}{\sqrt{6}}\right)^2$	$\lvert\varphi_L\rangle$
$+a$	$\left(\frac{1-\sqrt{2}}{\sqrt{6}}\right)^2$	$\lvert\varphi_R\rangle$
	$\overline{}\;+$ 1	

g) The state is $|\varphi_L\rangle = \frac{1}{\sqrt{2}}\left(|\varphi_g\rangle + |\varphi_e\rangle\right)$

since $|\varphi_g\rangle = \frac{1}{\sqrt{2}}\left(|\varphi_L\rangle + |\varphi_R\rangle\right)$ and $|\varphi_e\rangle = \frac{1}{\sqrt{2}}\left(|\varphi_L\rangle - |\varphi_R\rangle\right)$

h) $\langle\varphi_g|\hat{A}|\varphi_g\rangle = \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}-a & 0\\ 0 & a\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix} = 0$ $\Rightarrow$ expectation value for position is zero for system in state $|\varphi_g\rangle$

$\langle\varphi_e|\hat{A}|\varphi_e\rangle = \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}-a & 0\\ 0 & a\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{pmatrix} = 0$ $\Rightarrow$ expectation value for position is zero for system in state $|\varphi_e\rangle$

$\langle\varphi_g|\hat{A}|\varphi_e\rangle = \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}-a & 0\\ 0 & a\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{pmatrix} = -a$

$\langle\varphi_e|A|\varphi_g\rangle = \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}-a & 0\\ 0 & a\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix} = -a$

When the system is in a superposition of $|\varphi_g\rangle$ and $|\varphi_e\rangle$ the expectation value for position can be different from zero.

See answer i)

The position can oscillate with an amplitude $\propto \langle\varphi_g|\hat{A}|\varphi_e\rangle$ when in a superposition.

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i) State at $t=0$ denoted as $|\psi_0\rangle = |\varphi_L\rangle = \frac{1}{\sqrt{2}}\left(|\varphi_g\rangle + |\varphi_e\rangle\right)$

For investigating time evolution of $\hat{A}$, describe the state of the system as a superposition of energy eigen states.

$$\langle\hat{A}(t)\rangle = \langle\psi(t)|\hat{A}|\psi(t)\rangle = \langle\psi_0|\hat{U}^{+}\hat{A}\hat{U}|\psi_0\rangle$$

with $\hat{U} = e^{-\frac{i}{\hbar}\hat{H}t}$ $\Rightarrow$

$$\langle \hat{A}(t) \rangle = \frac{1}{2}\left(\langle \varphi_g| + \langle \varphi_e|\right) \hat{U}^{+} \hat{A}\, \hat{U} \left(|\varphi_g\rangle + |\varphi_e\rangle\right)$$

$$= \frac{1}{2}\left(e^{+i\omega_g t} \langle \varphi_g| + e^{+i\omega_e t} \langle \varphi_e|\right) \hat{A} \left(e^{-i\omega_g t} |\varphi_g\rangle + e^{-i\omega_e t} |\varphi_e\rangle\right)$$

$$= \frac{1}{2}\left(\langle \varphi_g|\hat{A}|\varphi_g\rangle + \langle \varphi_e|\hat{A}|\varphi_e\rangle + e^{+i(\omega_g - \omega_e)t} \langle \varphi_g|\hat{A}|\varphi_e\rangle + e^{+i(\omega_e - \omega_g)t} \langle \varphi_e|\hat{A}|\varphi_g\rangle\right)$$

$$= \frac{1}{2}\left(0 + 0 + e^{-i(\omega_e - \omega_g)t}(-a) + e^{+i(\omega_e - \omega_g)t}(-a)\right)$$

$$= -\frac{1}{2} a \cdot 2 \cos\left((\omega_e - \omega_g)\, t\right)$$

$$= -a \cos\left((\omega_e - \omega_g)\, t\right)$$

Where we used $\omega_e = \frac{E_e}{\hbar}$ and $\omega_g = \frac{E_g}{\hbar}$

$$E_e - E_g = \underbrace{-2T}_{>0} \Rightarrow$$

$$\langle \hat{A}(t) \rangle = -a \cos\left(\frac{|2T|}{\hbar} \cdot t\right)$$

The system oscillates between the two wells, from position $-a$ to $+a$ and back, and starts (as it should) indeed at $-a$ for $t=0$.

The angular frequency of the oscillations is $\frac{E_e - E_g}{\hbar} = \frac{|2T|}{\hbar}$

Problem W4.2

2a) For $n = 2,4,6\ldots\ldots$ $\varphi_n(x)$ is odd since

$$\hat{P}\varphi_n(x) = \hat{P}\left(\sqrt{\tfrac{2}{a}}\sin\left(\tfrac{n\pi x}{a}\right)\right) = \sqrt{\tfrac{2}{a}}\sin\left(-\tfrac{n\pi x}{a}\right) = -\sqrt{\tfrac{2}{a}}\sin\left(\tfrac{n\pi x}{a}\right) = -\varphi_n(x)$$

For $n = 1,3,5,7,\ldots$ $\varphi_n(x)$ is even since

$$\hat{P}\varphi_n(x) = \hat{P}\left(\sqrt{\tfrac{2}{a}}\cos\left(\tfrac{n\pi x}{a}\right)\right) = \sqrt{\tfrac{2}{a}}\cos\left(-\tfrac{n\pi x}{a}\right) = +\sqrt{\tfrac{2}{a}}\cos\left(\tfrac{n\pi x}{a}\right) = +\varphi_n(x)$$

2b) $\hat{D} = D(x) = eX$

$$\hat{P}(\hat{D}) = \hat{P}(D(x)) = D(-x) = -eX = -D(x) = -\hat{D}$$

So, $\hat{D}$ is an odd operator.

2c) Dipole oscillations are described by

$$\langle\psi(t)|\hat{D}|\psi(t)\rangle = \langle\hat{D}(t)\rangle$$

For the system in a state $\alpha|\varphi_m\rangle + \beta|\varphi_n\rangle$ at some time t this gives

$$\begin{aligned}\langle\hat{D}(t)\rangle = {} & \alpha^*\alpha\langle\varphi_m|\hat{D}|\varphi_m\rangle + \beta^*\beta\langle\varphi_n|\hat{D}|\varphi_n\rangle \\ & + e^{-\frac{i}{\hbar}(E_n-E_m)t}\,\alpha^*\beta\langle\varphi_m|\hat{D}|\varphi_n\rangle \\ & + e^{-\frac{i}{\hbar}(E_m-E_n)t}\,\alpha\beta^*\langle\varphi_n|\hat{D}|\varphi_m\rangle\end{aligned}$$

The oscillating terms are governed (in amplitude) by $\langle\varphi_n|\hat{D}|\varphi_m\rangle$ and $\langle\varphi_m|\hat{D}|\varphi_n\rangle = \langle\varphi_n|\hat{D}|\varphi_m\rangle^*$ (often called matrix elements), which are equal to

$$\langle\varphi_n|\hat{D}|\varphi_m\rangle = \int_{-a/2}^{a/2}\varphi_n^*(x)\,\hat{D}\,\varphi_m(x)\,dx$$

(note that this integration domain is symmetric around 0).

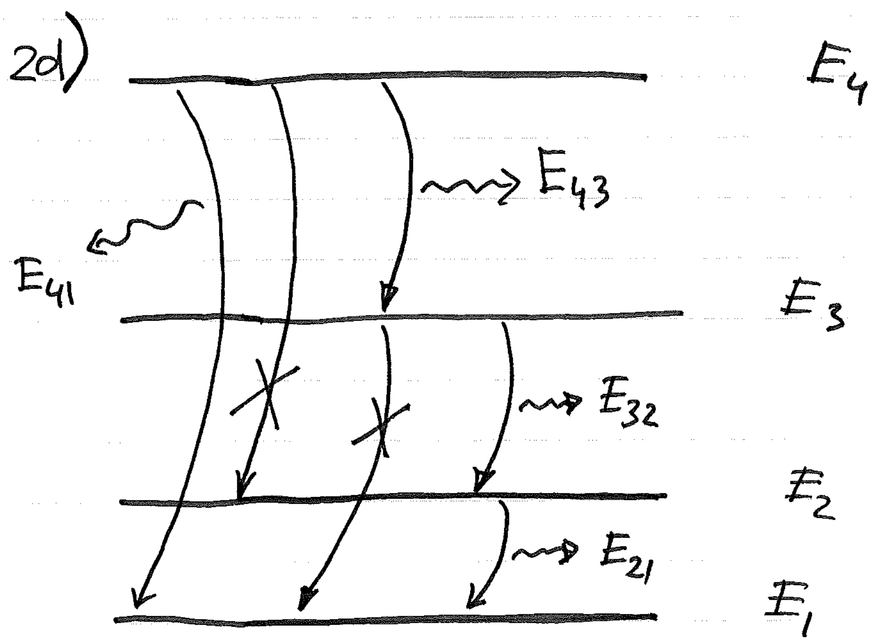
Here the $\varphi_n(x)$ and $\varphi_m(x)$ all have even or odd parity (symmetric or anti-symmetric in x).

The dipole operator $\hat{D} = e\hat{x}$, which is in x-representation $\hat{D} = D(x) = ex$, and is anti-symmetric in x.

Thus, integrals of the type $\int_{-a/2}^{a/2} \varphi_n^*(x)\, ex\, \varphi_m(x)\, dx$

yield zero if $\varphi_n(x)$ and $\varphi_m(x)$ are both even or both odd. Then, the system's dipole oscillations have zero amplitude and cannot emit a photon.

2d)



Transitions marked with X will not occur, since they are parity forbidden (photon cannot be emitted)

For the other transitions, the energy of the emitted photon is

$$\hbar\omega_{nm} = E_n - E_m = E_{nm}$$

So, the system can thus relax as follows:

1. Directly from E_4 to E_1, by emitting a photon with energy $\hbar\omega_{41} = E_4 - E_1$
   Final state is $\varphi_1(x)$

2. From E_4 to E_2 level is parity forbidden

3. From E_4 to E_3 level (under emission of a photon with energy $\hbar\omega_{43} = E_4 - E_3$), and then from E_3 to E_2 (directly to E_1 is now forbidden) by emitting a photon $\hbar\omega_{32} = E_3 - E_2$, and then from E_2 to E_1 by emitting a photon $\hbar\omega_{21} = E_2 - E_1$. Final state is $\varphi_1(x)$.

Problem W 4.3

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a) $\bullet\!\longrightarrow$ particle moves with **momentum** p, and passes some location at time t_0.

If the moment of passing this point is uncertain by an amount Δt, it corresponds to a wave packet at speed p/m that has a spread in position Δx, which gives

$$\Delta x = \left(\frac{p}{m}\right)\Delta t \Rightarrow \Delta t = \Delta x\left(\frac{m}{p}\right)$$

$$E = \frac{p^2}{2m} \Rightarrow dE = d\left(\frac{p^2}{2m}\right) = \frac{p}{m}dp \Rightarrow$$

$$\Delta E \approx \left(\frac{p}{m}\right)\Delta p \quad \Rightarrow \Delta p = \left(\frac{m}{p}\right)\Delta E$$

$$\Rightarrow \Delta x \Delta p = \left(\frac{p}{m}\right)\Delta t \cdot \left(\frac{m}{p}\right)\Delta E = \Delta E \Delta t \geq \frac{\hbar}{2}$$

b) $\lambda = \lambda_0 \pm \Delta\lambda = 800\text{ nm} \pm 20\text{ nm} \Rightarrow$

$\Delta\lambda = 20\text{ nm}$

$$E = hf = \frac{hc}{\lambda} \Rightarrow dE = d\left(\frac{hc}{\lambda}\right) \Rightarrow \Delta E \approx \left|\frac{hc}{\lambda_0^2}\Delta\lambda\right|$$

with $\lambda_0 = 800\text{ nm}$ $\quad h = 6.626\cdot 10^{-34}\text{ Js}$ $\quad c = 3\cdot 10^8\text{ m/s}$

$$\Rightarrow \Delta t = \frac{\hbar}{2\Delta E} = \frac{\hbar\lambda_0^2}{2hc\,\Delta\lambda} = \frac{\lambda_0^2}{4\pi c\,\Delta\lambda} = 8.5\text{ fs}$$

c) $\Delta t = 10\ ps$

Using the result of b),

$$\Delta E = \frac{\hbar}{2\Delta t} \Rightarrow \Delta E = 0.53 \cdot 10^{-23} J = 32\ \mu eV$$

$$\Delta\lambda = \frac{\lambda_0^2}{4\pi c\,\Delta t} \Rightarrow \Delta\lambda = 0.017\ nm$$

W 4.4.

a) $\hat{B}\hat{A}\varphi_1 = a\hat{B}\varphi_1 = \hat{A}\hat{B}\varphi_1$. So $\hat{B}\varphi_1$ is an eigenfunction of $\hat{A}$ with eigenvalue a. There are no linearly independent functions (apart from φ_1) also having eigenvalue a.

Hence $\hat{B}\varphi_1$ is ~~a~~ multiple of φ_1: $B\varphi_1 = \alpha\varphi_1$. This shows φ_1 is an eigenvalue of B.

b) $\hat{H} = \frac{p^2}{2m}$

$[H, p] = \frac{1}{2m}[p^2, p] = 0$

c) We look for functions of the form e^{ikx}.

$\sin(kx) = \frac{e^{ikx} - e^{-ikx}}{2i}$

$\cos(kx) = \frac{e^{ikx} + e^{-ikx}}{2}$

$\varphi_+ = \cos(kx) + i\sin(kx) = e^{+ikx}$

$\varphi_- = \cos(kx) - i\sin(kx) = e^{-ikx}$

φ_+ & φ_- are eigenfunctions of $-i\hbar\frac{\partial}{\partial x}$ with eigenvalue $\pm\hbar k$

3.7 $\hat{Q} f(x) = q f(x)$
$\hat{Q} g(x) = q g(x)$

a) $\hat{Q}\left[A f(x) + B g(x)\right] = A \hat{Q} f(x) + B \hat{Q} g(x) =$
$= A q f(x) + B q g(x) = q\left[A f(x) + B g(x)\right]$ ✓

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b) $f(x) = e^{x}$ $\quad g(x) = e^{-x}$

$$\frac{d^2 f(x)}{dx^2} = \frac{d^2 e^{x}}{dx^2} = 1 \frac{de^{x}}{dx} = 1^2 e^{x} = 1 f(x)$$

$$\frac{d^2 g(x)}{dx^2} = \frac{d^2 e^{-x}}{dx^2} = (-1) \frac{de^{-x}}{dx} = (-1)^2 e^{-x} = 1 g(x)$$

$$\alpha(x) = \frac{e^{x} + e^{-x}}{2} = \frac{f(x) + g(x)}{2} = \cosh(x)$$ even function

$$\beta(x) = \frac{e^{x} - e^{-x}}{2} = \frac{f(x) - g(x)}{2} = \sinh(x)$$ odd function

Since $\alpha(x)$ and $\beta(x)$ have different parity they are orthogonal.

~~3.10~~
3.8

a) looking at eqn. 3.29. we can say they are real.

Lets say. for two eigen functions.

$f = A_q e^{-iq\phi}$ & $g = A_{q'} e^{-iq'\phi}$ we have,

$$\langle f|g\rangle = A_q^* A_{q'} \int_0^{2\pi} e^{iq\phi} e^{-iq'\phi} d\phi$$

$$= A_q^* A_{q'} \left. \frac{e^{i(q-q')\phi}}{i(q-q')} \right|_0^{2\pi}$$

but. q & q' ar integers which forces $e^{i(q-q')2\pi}$ to 1

& hence $\langle f|g\rangle = 0$.

remember $q \neq q'$ otherwise denominator goes '0'

b) in problem 3.6. eigen values are real.

for any two eigenfunctions.

$f = A_q e^{\pm in\phi}$ & $g = A_{q'} e^{\pm in'\phi}$

$$\langle f|g\rangle = A_q^* A_{q'} \int_0^{2\pi} e^{\pm in\phi} e^{\mp in'\phi} d\phi$$

$$= A_q^* A_{q'} \left. \frac{e^{\pm i(n'-n)\phi}}{\pm i(n'-n)} \right|_0^{2\pi}$$

$$= \frac{A_q^* A_{q'}}{\pm i(n'-n)} \left[e^{\pm i(n'-n)2\pi} - 1 \right] = 0$$

3.10 $\Psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a} x\right)$ for $0 \leq x \leq a$

$$\hat{p}\,\Psi_1(x) = \frac{\hbar}{i}\frac{\partial}{\partial x}\sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right) = \frac{\hbar}{i}\sqrt{\frac{2}{a}}\,\frac{\pi}{a}\cos\left(\frac{\pi x}{a}\right) =$$

$$= \frac{\hbar}{i}\frac{\pi}{a}\frac{\cos\left(\frac{\pi x}{a}\right)}{\sin\left(\frac{\pi x}{a}\right)} \cdot \sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right) = \frac{\pi\hbar}{ia}\,\mathrm{cotg}\left(\frac{\pi x}{a}\right)\Psi_1(x)$$

$$\hat{p}\,\Psi_1(x) = f(x)\,\Psi_1(x)$$

Since $\hat{p}\,\Psi_1(x)$ is not igual to a constant multiple of $\Psi_1(x)$, $\Psi_1(x)$ is <u>not</u> an eigenfunction of $\hat{p}$.

3.12 $\langle x\rangle = \int_{-\infty}^{\infty} \Psi^* x \Psi \, dx$

But $\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} \phi(p,t)\, e^{\frac{ipx}{\hbar}}\, dp$ { This is the Fourier Transform, but with "p" instead of "k". Eq 3.55

$$\Rightarrow \langle x\rangle = \int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\phi^*(p'',t)\,e^{-\frac{ip''x}{\hbar}}\,dp''\right]\times\left[\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\phi(p',t)\,e^{\frac{ip'x}{\hbar}}\,dp'\right]dx$$

We have $x\int_{-\infty}^{\infty} e^{\frac{ip'x}{\hbar}}\phi\, dp' = \int_{-\infty}^{\infty} x e^{\frac{ip'x}{\hbar}}\phi\, dp' = \int_{-\infty}^{\infty}\phi\,(-i\hbar)\frac{de^{\frac{ip'x}{\hbar}}}{dp'}\,dp'$

Integrating by parts: $(-i\hbar)\int_{-\infty}^{\infty}\phi\,\frac{de^{\frac{ip'x}{\hbar}}}{dp'}\,dp' = (-i\hbar)\left[\underbrace{\phi e^{\frac{ip'x}{\hbar}}\Big|_{-\infty}^{\infty}}_{\text{this term is zero due to the boundary conditions}} - \int_{-\infty}^{\infty} e^{\frac{ip'x}{\hbar}}\frac{\partial\phi}{\partial p'}\,dp'\right]$

So $x\int_{-\infty}^{\infty} e^{\frac{ip'x}{\hbar}}\phi\, dp' = i\hbar\int_{-\infty}^{\infty} e^{\frac{ip'x}{\hbar}}\frac{\partial\phi}{\partial p'}\,dp'$

$$\Rightarrow \langle x\rangle = \frac{1}{2\pi\hbar}\iiint\left\{\phi^*(p'',t)\,e^{-\frac{ip''x}{\hbar}}\,e^{\frac{ip'x}{\hbar}}\,(i\hbar)\frac{\partial\phi(p',t)}{\partial p'}\right\}dp'\,dp''\,dx$$

Integrating in x

$\frac{1}{2\pi\hbar}\int e^{-i(p'-p'')\frac{x}{\hbar}}\,dx = \frac{1}{2\pi}\int e^{iaw}\,dw$ with $\begin{cases} w \equiv \frac{x}{\hbar} \Rightarrow dx = \hbar\, dw \\ a \equiv (p''-p') \end{cases}$

But $\frac{1}{2\pi}\int e^{ikx}\,dk = \delta(x)$ [See eq 2.144 page 77]

$$\Rightarrow \frac{1}{2\pi\hbar}\int e^{-i(p'-p'')\frac{x}{\hbar}}\,dx = \delta(p'-p'')$$

$$\Rightarrow \langle x\rangle = \iint\left\{\phi^*(p'',t)\left[-\frac{\hbar}{i}\frac{\partial\phi(p',t)}{\partial p'}\right]\delta(p'-p'')\right\}dp'\,dp''$$

$$\therefore \langle x\rangle = \int \phi^*(p,t)\left(-\frac{\hbar}{i}\frac{\partial}{\partial p}\right)\phi(p,t)\,dp$$

Note that while $\hat{x} = x$ and $\hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}$ in x-space, in p-space we have $\hat{p} = p$ and $\hat{x} = -\frac{\hbar}{i}\frac{\partial}{\partial p}$. (Eq 3.58)

(3.13) a) $[AB,C] = ABC - CAB = ABC - ACB + ACB - CAB$

$$[AB,C] = A(BC - CB) + (AC - CA)B = A[B,C] + [A,C]B$$

b) $[x^n,\hat{p}] = [x^{n-1}x,\hat{p}] = x^{n-1}[x,\hat{p}] + [x^{n-1},\hat{p}]x =$

$= x^{n-1}(i\hbar) + [x^{n-2}x,\hat{p}]x = x^{n-1}(i\hbar) + x^{n-2}[x,\hat{p}]x + [x^{n-2},\hat{p}]x^2 =$

$= x^{n-1}(i\hbar) + x^{n-1}(i\hbar) + [x^{n-3}x,\hat{p}]x^2$ and so on...

$\Rightarrow [x^n,\hat{p}] = n(i\hbar)x^{n-1}$

If you are not happy with this proof check by applying $[x^n,p]$ in a function f explicitly.
is in fact simpler

c) $[f(x),\hat{p}]\,g(x) = f(x)\left[(-i\hbar)\frac{\partial}{\partial x}g(x)\right] - (-i\hbar)\frac{\partial}{\partial x}\left[f(x)\,g(x)\right] =$

$= \cancel{(-i\hbar)f(x)\frac{\partial g(x)}{\partial x}} + i\hbar\left(\frac{\partial f(x)}{\partial x}\right)g(x) + i\hbar\, f(x)\frac{\partial g(x)}{\partial x} = i\hbar\left(\frac{\partial f(x)}{\partial x}\right)g(x)$

$\therefore\ [f(x),\hat{p}] = i\hbar\frac{\partial f(x)}{\partial x}$ //

3.14 $\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2$

For $\hat{A} = \hat{x}$ and $\hat{B} = \hat{H} = \frac{\hat{p}^2}{2m} + V$.

$$[\hat{x}, \hat{H}] = \left[\hat{x}, \frac{\hat{p}^2}{2m} + V\right] = \frac{1}{2m}[\hat{x}, \hat{p}^2] + [\hat{x}, V]$$

$$[\hat{x}, \hat{p}^2] = [\hat{x}, \hat{p}\hat{p}] = \hat{x}\hat{p}\hat{p} - \hat{p}\hat{p}\hat{x} = \hat{x}\hat{p}\hat{p} - \hat{p}\hat{x}\hat{p} + \hat{p}\hat{x}\hat{p} - \hat{p}\hat{p}\hat{x} =$$
$$= [\hat{x}, \hat{p}]\hat{p} + \hat{p}[\hat{x}, \hat{p}] = i\hbar\hat{p} + \hat{p}\, i\hbar = 2i\hbar\hat{p}$$

$[\hat{x}, V] = 0$ since $V = V(x)$.

$$\Rightarrow [\hat{x}, \hat{H}] = \frac{i\hbar}{m}\hat{p}$$

$$\frac{1}{2i}\langle [\hat{x}, \hat{H}] \rangle = \frac{1}{2i}\frac{i\hbar}{m}\langle \hat{p} \rangle = \frac{\hbar}{2m}\langle \hat{p} \rangle$$

$$\therefore \boxed{\sigma_x \sigma_H \geq \frac{\hbar}{2m} |\langle \hat{p} \rangle|}$$

For stationary states we have $\langle \hat{p} \rangle = 0$ and $\sigma_H = 0$. So from our equation we have $0 \geq 0$.

Note: for solving this you could also have used the rule

$$[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}$$

(3.15) $\left[\hat{P}, \hat{Q}\right] \neq 0$

Let $\{f_n\}$ be a complete set of eigenfunctions for <u>both</u> $\hat{P}$ and $\hat{Q}$:

$$\hat{P} f_n = \alpha_n f_n \qquad \text{and} \qquad \hat{Q} f_n = \beta_n f_n$$

Now we calculate $[\hat{P}, \hat{Q}] f_n$:

$$[\hat{P}, \hat{Q}] f_n = \hat{P}\hat{Q} f_n - \hat{Q}\hat{P} f_n = \beta_n \hat{P} f_n - \alpha_n \hat{Q} f_n = \beta_n \alpha_n f_n - \alpha_n \beta_n f_n$$

$\Rightarrow [\hat{P}, \hat{Q}] f_n = 0$ So if $[\hat{P}, \hat{Q}] \neq 0$ $f_n = 0$

(3.17) For all cases we have $\frac{\partial Q}{\partial t} = 0$ (assuming that $\hat{H}$ is time-independent).

So eq 3.71 reads: $\frac{d}{dt}\langle Q \rangle = \frac{i}{\hbar}\langle [\hat{H}, \hat{Q}] \rangle$

a) $\hat{Q} = 1$: $[\hat{H}, 1] = 0 \Rightarrow \frac{d}{dt}\langle \Psi | \Psi \rangle = 0$

b) $\hat{Q} = \hat{H}$: $[\hat{H}, \hat{H}] = 0 \Rightarrow \frac{d}{dt}\langle \Psi | \hat{H} | \Psi \rangle = \frac{d}{dt} E = 0$

c) $\hat{Q} = \hat{x}$: $[\hat{H}, \hat{x}] = -\frac{i\hbar}{m}\hat{p}$ (check solution for 3.14)

$$\frac{d}{dt}\langle x \rangle = \frac{1}{m}\langle \hat{p} \rangle$$

d) $\hat{Q} = \hat{p}$: $[\hat{H}, \hat{p}] = \left[-\frac{\hbar^2}{2m}\hat{p}^2 + V(x), \hat{p}\right] = -\frac{\hbar^2}{2m}\cancel{[\hat{p}^2, \hat{p}]} + [V(x), \hat{p}]$

$[\hat{H}, \hat{p}] = [V(x), \hat{p}] = i\hbar \frac{\partial V(x)}{\partial x}$ (problem 3.13 c)

$$\Rightarrow \frac{d}{dt}\langle \hat{p} \rangle = -\left\langle \frac{\partial V}{\partial x} \right\rangle$$

Letters (a) and (b) show the conservation of normalization and energy for stationary states. Letter (c) is the semi-classical formula $\langle \hat{p} \rangle = m\langle \dot{\hat{x}} \rangle$. (d) is the Ehrenfest's theorem (eq 1.38)

(3.27) $\hat{A}|\psi_1\rangle = a_1|\psi_1\rangle$ and $\hat{A}|\psi_2\rangle = a_2|\psi_2\rangle$

$\hat{B}|\phi_1\rangle = b_1|\phi_1\rangle$ and $\hat{B}|\phi_2\rangle = b_2|\phi_2\rangle$

$$|\psi_1\rangle = \frac{1}{5}\left(3|\phi_1\rangle + 4|\phi_2\rangle\right)$$

$$|\psi_2\rangle = \frac{1}{5}\left(4|\phi_1\rangle - 3|\phi_2\rangle\right)$$

So $|\phi_1\rangle = \frac{1}{5}\left(3|\psi_1\rangle + 4|\psi_2\rangle\right)$ and $|\phi_2\rangle = \frac{1}{5}\left(4|\psi_1\rangle - 3|\psi_2\rangle\right)$

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a) The state of the system goes to $|\psi_1\rangle$ if a_1 is measured.

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b) If B is measured we get b_1 with probability $\left|\frac{3}{5}\right|^2 = \frac{9}{25}$ and b_2 with $\left|\frac{4}{5}\right|^2 = \frac{16}{25}$.

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c) After measuring B we have the probabilities:

- $9/25$ of the system being in state $|\phi_1\rangle$
- $16/25$ of the system being in state $|\phi_2\rangle$

The probability of getting a_1 in the first case is $\left|\frac{3}{5}\right|^2 = \frac{9}{25}$ and in the second is $\left|4/5\right|^2 = 16/25$. So the total probability is:

$$\frac{9}{25}\cdot\frac{9}{25} + \frac{16}{25}\cdot\frac{16}{25} = \frac{81+256}{625} = \frac{337}{625} = 0.5392$$

3.31 Eq. 3.71 says $\frac{d}{dt}\langle\hat{x}\hat{p}\rangle = \frac{i}{\hbar}\left\langle[\hat{H},\hat{x}\hat{p}]\right\rangle + \left\langle\frac{\partial}{\partial t}(\hat{x}\hat{p})\right\rangle$

Since $\hat{x}$ and $\hat{p}$ are time independent $\langle\frac{\partial}{\partial t}(\hat{x}\hat{p})\rangle = 0$

$[\hat{H},\hat{x}\hat{p}] = [\hat{H},\hat{x}]\hat{p} + \hat{x}[\hat{H},\hat{p}]$ From 3.14 and 3.17:

$[\hat{H},\hat{x}] = -\frac{i\hbar}{m}\hat{p}$ and $[\hat{H},\hat{p}] = i\hbar\frac{\partial V(x)}{\partial x}$. So:

$$\frac{d}{dt}\langle\hat{x}\hat{p}\rangle = \frac{i}{\hbar}(i\hbar)\left\langle\left(-\frac{\hat{p}^2}{m}\right)\frac{2}{2} + \hat{x}\frac{\partial V}{\partial x}\right\rangle = 2\left\langle\frac{\hat{p}^2}{2m}\right\rangle - \left\langle\hat{x}\frac{dV}{dx}\right\rangle$$

$$\therefore \frac{d}{dt}\langle\hat{x}\hat{p}\rangle = 2\langle\hat{T}\rangle - \left\langle x\frac{dV}{dx}\right\rangle$$

For a stationary state every expectation value is constant in time (equation 2.9). So:

$$2\langle\hat{T}\rangle = \left\langle x\frac{dV}{dx}\right\rangle$$

For the harmonic oscilator $V = \frac{m\omega^2x^2}{2} \Rightarrow \frac{dV}{dx} = m\omega^2 x$. So:

$$2\langle\hat{T}\rangle = \langle x \cdot m\omega^2 x\rangle = \left\langle 2\frac{m\omega^2x^2}{2}\right\rangle = 2\langle V\rangle$$

$$\therefore \langle\hat{T}\rangle = \langle V\rangle$$

(3.38) $H = \begin{pmatrix} \hbar\omega & 0 & 0 \\ 0 & 2\hbar\omega & 0 \\ 0 & 0 & 2\hbar\omega \end{pmatrix}$; $A = \begin{pmatrix} 0 & \lambda & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 2\lambda \end{pmatrix}$; $B = \begin{pmatrix} 2\mu & 0 & 0 \\ 0 & 0 & \mu \\ 0 & \mu & 0 \end{pmatrix}$;

a) Since H is diagonal, the eigenvalues are:

$$\boxed{\epsilon_1 = \hbar\omega \quad \text{and} \quad \epsilon_2 = \epsilon_3 = 2\hbar\omega}$$

And the eigenvectors are:

$$\boxed{|h_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad |h_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad |h_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$$

Now for A:

$$\begin{vmatrix} -a & \lambda & 0 \\ \lambda & -a & 0 \\ 0 & 0 & 2\lambda - a \end{vmatrix} = a^2(2\lambda - a) - \lambda^2(2\lambda - a) = (2\lambda - a)(a^2 - \lambda^2) = 0$$

So: $\boxed{a_1 = 2\lambda \; ; \; a_2 = \lambda \; ; \; a_3 = -\lambda}$

$$\begin{pmatrix} 0 & \lambda & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 2\lambda \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = a \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \Rightarrow \begin{cases} \lambda\beta = a\alpha \\ \lambda\alpha = a\beta \\ 2\lambda\gamma = a\gamma \end{cases}$$

$\underline{a = a_1 = 2\lambda} \Rightarrow \begin{cases} \lambda\beta = 2\lambda\alpha \Rightarrow \beta = 2\alpha \\ \lambda\alpha = 2\lambda\beta \Rightarrow \alpha = 2\beta \\ 2\lambda\gamma = 2\lambda\gamma \end{cases} \Rightarrow \alpha = \beta = 0$ $\quad \boxed{|a_1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$

$\underline{a = a_2 = \lambda} \Rightarrow \begin{cases} \lambda\beta = \lambda\alpha \Rightarrow \beta = \alpha \\ \lambda\alpha = \lambda\beta \Rightarrow \alpha = \beta \\ 2\lambda\gamma = \lambda\gamma \Rightarrow \gamma = 0 \end{cases} \Rightarrow \boxed{|a_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}$

$\underline{a = a_3 = -\lambda} \Rightarrow \begin{cases} \lambda\beta = -\lambda\alpha \Rightarrow \beta = -\alpha \\ \lambda\alpha = -\lambda\beta \Rightarrow \alpha = -\beta \\ 2\lambda\gamma = -\lambda\gamma \Rightarrow \gamma = 0 \end{cases} \Rightarrow \boxed{|a_3\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}$

Writing as a function of $|h_n\rangle$:

$$\boxed{|a_1\rangle = |h_3\rangle \; ; \; |a_2\rangle = \frac{|h_1\rangle + |h_2\rangle}{\sqrt{2}} \; ; \; |a_3\rangle = \frac{|h_1\rangle - |h_2\rangle}{\sqrt{2}}}$$

For B:

$$\begin{vmatrix} 2\mu-b & 0 & 0 \\ 0 & -b & \mu \\ 0 & \mu & -b \end{vmatrix} = b^2(2\mu-b) - \mu^2(2\mu-b) = (2\mu-b)(b^2-\mu^2) = 0$$

So: $\boxed{b_1 = 2\mu \;;\quad b_2 = \mu \;;\quad b_3 = -\mu}$

$$\begin{pmatrix} 2\mu & 0 & 0 \\ 0 & 0 & \mu \\ 0 & \mu & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = b\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \Rightarrow \begin{cases} 2\mu\alpha = b\alpha \\ \mu\gamma = b\beta \\ \mu\beta = b\gamma \end{cases}$$

$\underline{b = b_1 = 2\mu}$: $\begin{cases} 2\mu\alpha = 2\mu\alpha \\ \mu\gamma = 2\mu\beta \Rightarrow \gamma = 2\beta \\ \mu\beta = 2\mu\gamma \Rightarrow \beta = 2\gamma \end{cases} \Rightarrow \beta = \gamma = 0 \Rightarrow \boxed{|b_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$

$\underline{b = b_2 = \mu}$: $\begin{cases} 2\mu\alpha = \mu\alpha \\ \mu\gamma = \mu\beta \\ \mu\beta = \mu\gamma \end{cases} \Rightarrow \begin{cases} \alpha = 0 \\ \gamma = \beta \end{cases}$ $\quad \boxed{|b_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}$

$\underline{b = b_3 = -\mu}$: $\begin{cases} 2\mu\alpha = -\mu\alpha \\ \mu\gamma = -\mu\beta \\ \mu\beta = -\mu\gamma \end{cases} \Rightarrow \begin{cases} \alpha = 0 \\ \gamma = -\beta \end{cases}$ $\quad \boxed{|b_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}}$

Or:

$$\boxed{|b_1\rangle = |h_1\rangle \;;\quad |b_2\rangle = \frac{|h_2\rangle + |h_3\rangle}{\sqrt{2}} \;;\quad |b_3\rangle = \frac{|h_2\rangle - |h_3\rangle}{\sqrt{2}}}$$

---

b) $\langle H \rangle = \langle \mathcal{S}(0)|H|\mathcal{S}(0)\rangle = \begin{pmatrix} c_1^* & c_2^* & c_3^* \end{pmatrix}\begin{pmatrix} \hbar\omega & 0 & 0 \\ 0 & 2\hbar\omega & 0 \\ 0 & 0 & 2\hbar\omega \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$

$$\langle H \rangle = \hbar\omega\left[|c_1|^2 + 2|c_2|^2 + 2|c_3|^2\right]$$

$$\langle A \rangle = \langle \mathcal{S}(0)|A|\mathcal{S}(0)\rangle = \begin{pmatrix} c_1^* & c_2^* & c_3^* \end{pmatrix}\begin{pmatrix} 0 & \lambda & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 2\lambda \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$\langle A \rangle = \lambda\left[c_1^* c_2 + c_2^* c_1 + 2|c_3|^2\right]$$

$$\langle B\rangle = \langle \mathcal{S}(0)|B|\mathcal{S}(0)\rangle = \begin{pmatrix} c_1^* & c_2^* & c_3^* \end{pmatrix}\begin{pmatrix} 2\mu & 0 & 0 \\ 0 & 0 & \mu \\ 0 & \mu & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$\langle B\rangle = \mu\left[2|c_1|^2 + c_2^* c_3 + c_3^* c_2\right]$$

---

c)

$$|\Psi(t)\rangle = e^{-\frac{i\hat{H}t}{\hbar}}|\Psi(0)\rangle$$

$$|\mathcal{S}(0)\rangle = c_1|h_1\rangle + c_2|h_2\rangle + c_3|h_3\rangle$$

$$|\mathcal{S}(t)\rangle = c_1 e^{-\frac{i\hbar\omega t}{\hbar}}|h_1\rangle + c_2 e^{-\frac{i2\hbar\omega t}{\hbar}}|h_2\rangle + c_2 e^{-\frac{i2\hbar\omega t}{\hbar}}|h_3\rangle$$

$$\boxed{|\mathcal{S}(t)\rangle = e^{-i2\omega t}\left(e^{i\omega t}c_1|h_1\rangle + c_2|h_2\rangle + c_3|h_3\rangle\right)}$$

If you measure energy you get:

$h_1 = \hbar\omega$ with probability $|c_1|^2$ and $h_2 = h_3 = 2\hbar\omega$ probability $(|c_2|^2 + |c_3|^2)$.

If you measure A you get:

$a_1 = 2\lambda$ with probability $P_{a_1} = |\langle a_1|\mathcal{S}(t)\rangle|^2 = |\langle h_3|\mathcal{S}(t)\rangle|^2 = |e^{-i2\omega t}c_3|^2$

$$P_{a_1} = |c_3|^2$$

$a_2 = \lambda$ with probability $P_{a_2} = |\langle a_2|\mathcal{S}(t)\rangle|^2 = \left|\frac{1}{\sqrt{2}}\left(\langle h_1| + \langle h_2|\right)|\mathcal{S}(t)\rangle\right|^2$

$$P_{a_2} = \left|\frac{1}{\sqrt{2}}\left(e^{-i\omega t}c_1 + e^{-i2\omega t}c_2\right)\right|^2 = \frac{1}{2}\left(e^{i\omega t}c_1^* + e^{i2\omega t}c_2^*\right)\left(e^{-i\omega t}c_1 + e^{-2i\omega t}c_2\right)$$

$$P_{a_2} = \frac{1}{2}\left(|c_1|^2 + |c_2|^2 + e^{-i\omega t}c_1^* c_2 + e^{i\omega t}c_2^* c_1\right)$$

$a_3 = -\lambda$ with probability $P_{a_3} = |\langle a_3|\mathcal{S}(t)\rangle|^2 = \left|\frac{1}{\sqrt{2}}\left(\langle h_1| - \langle h_2|\right)|\mathcal{S}(t)\rangle\right|^2$

$$P_{a_3} = \left|\frac{1}{\sqrt{2}}\left(e^{-i\omega t}c_1 - e^{-i2\omega t}c_2\right)\right|^2 = \frac{1}{2}\left(e^{i\omega t}c_1^* - e^{2i\omega t}c_2^*\right)\left(e^{-i\omega t}c_1 - e^{-2i\omega t}c_2\right)$$

$$P_{a_3} = \frac{1}{2}\left(|c_1|^2 + |c_2|^2 - e^{-i\omega t}c_1^* c_2 - e^{i\omega t}c_2^* c_1\right)$$

If you measure B you get:

$\underline{b_1 = 2\mu}$ with probability $P_{b_1} = |\langle b_1 | \mathcal{S}(t)\rangle|^2 = |\langle h_1 | \mathcal{S}(t)\rangle|^2$

$$P_{b_1} = |c_1|^2$$

$\underline{b_2 = \mu}$ with probability $P_{b_2} = |\langle b_2 | \mathcal{S}(t)\rangle|^2 = \left|\frac{1}{\sqrt{2}}\left(\langle h_2| + \langle h_3|\right) | \mathcal{S}(t)\rangle\right|^2$

$$P_{b_2} = \left|\frac{1}{\sqrt{2}} e^{-2i\omega t}(c_2 + c_3)\right|^2 = \frac{1}{2}(c_2^* + c_3^*)(c_2 + c_3)$$

$$P_{b_2} = \frac{1}{2}\left(|c_2|^2 + |c_3|^2 + c_2^* c_3 + c_3^* c_2\right)$$

$\underline{b_3 = -\mu}$ with Probability $P_{b_3} = |\langle b_3 | \mathcal{S}(t)\rangle|^2 = \left|\frac{1}{\sqrt{2}}\left(\langle h_2| - \langle h_3|\right) | \mathcal{S}(t)\rangle\right|^2$

$$P_{b_3} = \left|\frac{1}{\sqrt{2}} e^{-2i\omega t}(c_2 + c_3)\right|^2 = \frac{1}{2}(c_2^* + c_3^*)(c_2 + c_3)$$

$$P_{b_3} = \frac{1}{2}\left(|c_2|^2 + |c_3|^2 - c_2^* c_3 - c_3^* c_2\right)$$