Problem set for the 6th week of the course Quantum Physics 1

For the tutorials sessions of 8 and 10 October 2014

Homework, to be made before the werkcollege:
From the book (Griffiths 2nd Ed.) Chapter 4 - 4.19, 4.27, 4.29, 4.31, 4.35.

Problems to work on during werkcollege:
Problem W6.1 – W6.5 (this hand out), and from the book Chapter 4 - 4.25, 4.32, 4.34.
This is the minimal set you need to do.
Other good problems that we selected (we advise you to make these for the topics where you need or like to do extra training): from the book Chapter 4 - 4.18.

Problem W6.1

If an electron is in a circular orbit around a nucleus, this state has an angular momentum $\boldsymbol{L}$ and the circular motion of the electron also causes a magnetic moment $\boldsymbol{\mu}$ (at least when considering it semi-classically, quantum mechanically these quantities can be zero for certain states). These two quantities are proportional to each other, according to $\boldsymbol{\mu} = \gamma\, \boldsymbol{L}$, where γ is the so-called gyromagnetic ratio. The purpose of this exercise is to obtain the value of $\gamma = -|e/2m_e|$ from a semi-classical calculation, where $-e$ is the electron charge and m_e the electron mass (but see the footnote in the book by Griffiths below Eq. [4.156] for the result of a full quantum relativistic calculation).

It turns out, that you do not need to know details of the electron orbit. For simplicity, you can therefore assume here that the electron is on a circular orbit of radius r with velocity v. Further use that the magnetic moment of a circulation current I in a loop of area A is $\mu = I\,A$. Use this to calculate γ.

Problem W6.2

Most large hospitals now have a system for *Magnetic Resonance Imaging* (MRI, Nobel prize 2003). This is an apparatus that can make images of soft tissues inside the human body in a very noninvasive manner. The imaging technique is based on resonantly driving and detecting the signals of the quantum dynamics of nuclear spins. It mainly uses the nuclear spins of hydrogen atoms. Contrast in the images appears because the spin dynamics of the hydrogen nucleus depends on the chemical environment of the hydrogen atom (very small but detectable deviations). One can in this way basically see which organic molecules (that contain hydrogen) are present at a certain place in the body, and also how much water molecules are present at a certain place. The same technique in a physics lab is not called MRI, but Nuclear Magnetic Resonance (NMR). The technique is based on the dynamics of spins in a magnetic field, called Larmor precession (see the book by Griffiths, Eqs. [4.164 – 4.167]). For electron spins, the technique is called ESR (Electron Spin Resonance) or EPR (Electron Paramagnetic Resonance).

A person in an MRI machine is brought to a place with a strong magnetic field. Assume for this problem that this is a field of strength $B_z = 2$ Tesla (typical value in practice, this is 40,000 times stronger than the Earth magnetic field). The field is applied in the z-direction. The spin of the hydrogen nucleus has $s = ½$. The Hamiltonian for this spin is now

$$\hat{H} = -\gamma\, B_z\, \hat{S}_z\,,$$

where is $\gamma = +267.5\cdot 10^6$ rad s^{-1} T^{-1}, and $\hat{S}_z$ is the operator for the z-component of the spin. The operators for the x- and y-component of the spin are $\hat{S}_x$ and $\hat{S}_y$, respectively. For notation use that the states $|\uparrow\rangle$ and $|\downarrow\rangle$ represent spin-up and spin-down along the z-axis.

a)

Which of the operators $\hat{S}_z$, $\hat{S}_x$ and $\hat{S}_y$ commute with the Hamiltonian, and which ones do not? Explain your answer.

b)
What are the energy eigenstates and energy eigenvalues of this Hamiltonian? In your answer, also specify which state is the ground state and which state or states is/are excited state(s).

c)
The MRI technique relies on applying an oscillating magnetic field that is resonant with the dynamics of the spin. At what oscillation frequency should one apply such an oscillating magnetic field in the MRI setup that we consider here? Calculate an actual number.

d)
Calculate (or simply list the answer if you know it) all matrix elements $\langle\uparrow|\hat{S}_j|\uparrow\rangle$, $\langle\uparrow|\hat{S}_j|\downarrow\rangle$, $\langle\downarrow|\hat{S}_j|\uparrow\rangle$ and $\langle\downarrow|\hat{S}_j|\downarrow\rangle$, for the three cases $j = x,y,z$.

e)
The spins of the hydrogen nuclei are at time t = 0 prepared in the state $|\Psi_0\rangle = \sqrt{\tfrac{2}{3}}|\uparrow\rangle + i\sqrt{\tfrac{1}{3}}|\downarrow\rangle$.
Calculate how the expectation values $\langle\hat{S}_x\rangle$, $\langle\hat{S}_y\rangle$ and $\langle\hat{S}_z\rangle$ depend on time for $t > 0$.
Hint: for short notation you could use the results $\langle\uparrow|\hat{S}_j|\uparrow\rangle$ etc. of question d).

f)
As follow up on question e), make graphs for $\langle\hat{S}_x\rangle$, $\langle\hat{S}_y\rangle$ and $\langle\hat{S}_z\rangle$ as a function of time for $t \geq 0$ (so, you must make 3 graphs). Describe in words what the dynamics is of the spins in question e). If you like you can make a drawing with your explanation.

g)
If you want to measure the time-dependent spin dynamics of such spins (for example the dynamics of question e) and f)), how could you measure that? Explain in some detail (10 lines of text) what the physical signal is that you would try to detect, and what apparatus you could build for that.

Problem W6.3
A certain atom is in a state with its total orbital angular momentum vector $\boldsymbol{L}$ (described by the operators $\hat{L}_x, \hat{L}_y, \hat{L}_z$, and $\hat{L}^2$) defined by orbital quantum number $l = 1$.

a) What is in this case the length of this vector for total angular momentum $\boldsymbol{L}$?

For the system in this state, the operator for the z-component of angular momentum is $\hat{L}_z$. It has three eigenvalues, $+\hbar$ (with corresponding eigenstate $|+_z\rangle$), $0\hbar$ (with eigenstate $|0_z\rangle$), and $-\hbar$ (with eigenstate $|-_z\rangle$). This operator can be represented as a matrix, and the ket-states as column vectors, using the basis spanned by $|+_z\rangle$, $|0_z\rangle$ and $|-_z\rangle$, according to

$$\hat{L}_z \leftrightarrow \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad |+_z\rangle \leftrightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |0_z\rangle \leftrightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \text{ and } \quad |-_z\rangle \leftrightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Using this same basis for the representation, the operator and eigenstates for the system's x-component of angular momentum are given by

$$\hat{L}_x \leftrightarrow \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad |+_x\rangle \leftrightarrow \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}, \qquad |0_x\rangle \leftrightarrow \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}, \text{ and } \qquad |-_x\rangle \leftrightarrow \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}.$$

b) *Calculate* with this information what the eigenvalues are that belong to $|+_x\rangle$, $|0_x\rangle$ and $|-_x\rangle$.

c) At some point the system is in the normalized state $|\Psi_1\rangle = \sqrt{\frac{1}{8}}\ |+_z\rangle + \sqrt{\frac{3}{8}}\,|0_z\rangle + \sqrt{\frac{4}{8}}\,|-_z\rangle$.
Calculate for this state the expectation value for angular momentum in z-direction and the expectation value for angular momentum in x-direction.

d) At some point the system is in the normalized state $|\Psi_2\rangle = \sqrt{\frac{1}{3}}\ |+_z\rangle + \sqrt{\frac{1}{3}}\,|0_z\rangle + \sqrt{\frac{1}{3}}\,|-_z\rangle$, and you are going to measure the x-component of the system's angular momentum. What are the possible measurement results? Calculate for each possible measurement result the probability given that the system is in state $|\Psi_2\rangle$.

e) At some point the system is in the state $|\Psi_3\rangle = 3i\ |+_z\rangle + 2\,|0_z\rangle - i\,|-_z\rangle$. Note that this state is not normalized. Calculate for this state $\langle \hat{L}_z \rangle$.

f) At some point the system is in the normalized state $|\Psi_4\rangle = \sqrt{\frac{1}{2}}\ |+_z\rangle + \sqrt{\frac{1}{2}}\,|-_z\rangle$. Calculate for this state the quantum uncertainty ΔL_z in the z-component of the system's angular momentum.

g) In this problem we study again how spins precess in a magnetic field. In problem W6.2 we studied this for spin ½ system. You will calculate it here for our spin 1 system.
Assume now the system is prepared in a different state (now again a superposition of eigenstates of $\hat{L}_z$), $|\Psi_5\rangle = \sqrt{\frac{1}{2}}\ |+_z\rangle + \sqrt{\frac{1}{2}}\,|0_z\rangle$, at time $t = 0$. Another change to the system is that one now applies an external magnetic field with magnitude B along the z-axis. The Hamiltonian of the system is now, $\hat{H} = \gamma B \hat{L}_z$, where γ is a constant that reflects how much the energy of angular momentum states shifts when applying the field. Calculate how the expectation value for angular momentum in x-direction depends on time.

Use Dirac notation and the time-evolution operator $\hat{U} = e^{\frac{-i\hat{H}t}{\hbar}}$ to get started and first describing the time dependence, but then switch to the matrix notation of this problem to work it out in detail.

Problem W6.4

This problem aims to help you learn how **addition of angular momentum** works out in quantum physics. This is very important for later on in your study. The rules are in fact quite simple. But remember that it is one step more complicated than only adding two integer numbers (a common mistake on the exam). The problem starts with some simple questions about individual angular momentums (electron spin and orbital angular momentum). Part b) is then about addition of angular momentum.

Consider a hydrogen atom. Assume that the electron is in a state for which the orbital wavefunction has the quantum numbers $n = 4$ and $l = 2$ (using the usual notation). In this problem you need to consider the electron as a particle that has spin.

a) We have an instrument that can be used to measure the z-component of angular momentum or to measure the length of an angular momentum vector. Before each measurement the electron is first prepared in the state with $n = 4$ and $l = 2$.

a1) The instrument is tuned to measure the z-component of the spin of the electron. What are the possible measurement outcomes?

a2) The instrument is tuned to measure the length of the angular momentum vector while only measuring on the spin of the electron. What are the possible measurement outcomes?

a3) The instrument is tuned to measure the z-component of the orbital angular momentum of the electron. What are the possible measurement outcomes?

a4) The instrument is tuned to measure the length of the angular momentum vector while only measuring on the orbital angular momentum of the electron. What are the possible measurement outcomes?

a5) A similar instrument can be used to measure the z-component of the magnetic dipole moment of the electron (from the electron on itself, it is not measuring a magnetic dipole moment from the orbital state). What are the possible measurement outcomes if this instrument is used?

b) Now, the same instrument as in questions **a1)-a4)** is tuned to measure the *total amount of angular momentum in the atom* (the addition of orbital and spin contributions). In this problem you are supposed to ignore the angular momentum of the nucleus. Assume again that before each measurement the electron is first prepared in the state with $n = 4$ and $l = 2$. What are now the possible measurement outcomes when:

b1) we measure the length of the angular momentum vector of the atom's total amount of angular momentum.

b2) we measure the z-component of the total amount of angular momentum.

Problem W6.5

This problem aims to help you learn the concept of **Clebsch-Gordan coefficients** in quantum physics. The hydrogen atom is a good system for getting insight in the role it plays in quantum physics, and for learning how the Clebsch-Gordan coefficients are used.

So, consider a hydrogen atom. To keep it somewhat simple, we only consider the spin $\vec{S}$ of the electron and the orbital angular momentum $\vec{L}$ of the electron (we neglect the spin of the nucleus). In a real hydrogen atom, the spin of the electron and the orbital angular momentum of the electron interact with each other since they act as two magnetic dipoles that apply a force on each other. This gives small shifts to the energy levels of the hydrogen atom (not discussed in Chapter 4 of the book). Also, it gives that some levels that are *degenerate* according to the model of Chapter 4, split into a few levels with slightly different energies.

You can see this as follows. The spin of the electron corresponds to a rotating charge, and therefore the electron has a magnetic dipole moment that is proportional to its spin (see problem W6.4a5). The same is true for the orbital angular momentum of the electron. In that case there is an orbiting charge that gives a magnetic dipole moment that is proportional to the orbital angular momentum (see problem W6.1). The interaction between them is therefore called spin-orbit interaction. You may think of this as the effect of the charge current in a loop (causing a magnetic field) that the electron feels because the positively charged nucleus flies around the electron (that's how it is from the electron's point of view).

It turns out that due to the spin-orbit interaction, the energy levels (energy eigenstates) of the atom correspond to states that are also eigenstates for the *total amount of angular momentum in the atom* $\vec{J} = \vec{L} + \vec{S}$ (specifically, the complete Hamiltonian commutes with the operator $\hat{J}^2$, and therefore also with the operator $\hat{J}_z$). So, when the atom is in an energy eigenstate and also in an eigenstate of $\hat{J}_z$, the state of the spin and the orbital angular momentum are then not in eigenstates of $\hat{S}_z$ and $\hat{L}_z$. Instead, the spin is then in a superposition of different spin eigenstates, and the same holds for the orbital angular momentum.

The question is now: how is this superposition exactly? It is important to know this, for example when you want to understand whether a laser can induce transitions between certain energy levels of the atom.

Let's introduce the notation for the quantum numbers as follows:

> The operators for $\vec{L}$ have quantum numbers l for its length and m_l for its z-component.
>
> The operators for $\vec{S}$ have quantum numbers s for its length, and m_s for its z-component.
>
> The operators for $\vec{J}$ have quantum numbers j for its length and m_j for its z-component.

a) Consider a state for which it is known that $l = 2$ and $s = ½$. The atom is in an energy eigenstate for which the total angular momentum is in the eigenstate

$$\left| j=\tfrac{3}{2}, m_j=+\tfrac{1}{2} \right\rangle = \alpha \left| l=2, m_l=+1 \right\rangle \left| s=\tfrac{1}{2}, m_s=-\tfrac{1}{2} \right\rangle + \beta \left| l=2, m_l=0 \right\rangle \left| s=\tfrac{1}{2}, m_s=+\tfrac{1}{2} \right\rangle .$$

Use the table in the book on p. 188 to determine the values of α and β.

b) With the atom in this state, we bring the atom into an apparatus that will measure the z-component of the electron spin. What is the probability that the measurement result is spin-up?