FROM:

Problem set for the 6th week of the course Quantum Physics 1

For the tutorials sessions of 8 and 10 October 2014

Homework, to be made before the werkcollege:
From the book (Griffiths 2nd Ed.) Chapter 4 - 4.19, 4.27, 4.29, 4.31, 4.35.

Problems to work on during werkcollege:
Problem W6.1 – W6.5 (this hand out), and from the book Chapter 4 - 4.25, 4.32, 4.34.
This is the minimal set you need to do.
Other good problems that we selected (we advise you to make these for the topics where you need or like to do extra training): from the book Chapter 4 - 4.18.

Problem W6.2

Most large hospitals now have a system for *Magnetic Resonance Imaging* (MRI, Nobel prize 2003). This is an apparatus that can make images of soft tissues inside the human body in a very noninvasive manner. The imaging technique is based on resonantly driving and detecting the signals of the quantum dynamics of nuclear spins. It mainly uses the nuclear spins of hydrogen atoms. Contrast in the images appears because the spin dynamics of the hydrogen nucleus depends on the chemical environment of the hydrogen atom (very small but detectable deviations). One can in this way basically see which organic molecules (that contain hydrogen) are present at a certain place in the body, and also how much water molecules are present at a certain place. The same technique in a physics lab is not called MRI, but Nuclear Magnetic Resonance (NMR). The technique is based on the dynamics of spins in a magnetic field, called Larmor precession (see the book by Griffiths, Eqs. [4.164 – 4.167]). For electron spins, the technique is called ESR (Electron Spin Resonance) or EPR (Electron Paramagnetic Resonance).

A person in an MRI machine is brought to a place with a strong magnetic field. Assume for this problem that this is a field of strength B_z = 2 Tesla (typical value in practice, this is 40,000 times stronger than the Earth magnetic field). The field is applied in the z-direction. The spin of the hydrogen nucleus has $s = ½$. The Hamiltonian for this spin is now

$$\hat{H} = -\gamma B_z \hat{S}_z ,$$

where is $\gamma = +267.5\cdot 10^6$ rad s^{-1} T^{-1}, and $\hat{S}_z$ is the operator for the z-component of the spin. The operators for the x- and y-component of the spin are $\hat{S}_x$ and $\hat{S}_y$, respectively. For notation use that the states $|\uparrow\rangle$ and $|\downarrow\rangle$ represent spin-up and spin-down along the z-axis.

a)

Which of the operators $\hat{S}_z$, $\hat{S}_x$ and $\hat{S}_y$ commute with the Hamiltonian, and which ones do not? Explain your answer.

b)

What are the energy eigenstates and energy eigenvalues of this Hamiltonian? In your answer, also specify which state is the ground state and which state or states is/are excited state(s).

c)

The MRI technique relies on applying an oscillating magnetic field that is resonant with the dynamics of the spin. At what oscillation frequency should one apply such an oscillating magnetic field in the MRI setup that we consider here? Calculate an actual number.

d)

Calculate (or simply list the answer if you know it) all matrix elements $\langle\uparrow|\hat{S}_j|\uparrow\rangle$, $\langle\uparrow|\hat{S}_j|\downarrow\rangle$, $\langle\downarrow|\hat{S}_j|\uparrow\rangle$ and $\langle\downarrow|\hat{S}_j|\downarrow\rangle$, for the three cases $j = x,y,z$.

e)

The spins of the hydrogen nuclei are at time $t = 0$ prepared in the state $\left|\Psi_0\right\rangle = \sqrt{\tfrac{2}{3}}\left|\uparrow\right\rangle + i\sqrt{\tfrac{1}{3}}\left|\downarrow\right\rangle$.

Calculate how the expectation values $\left\langle \hat{S}_x \right\rangle$, $\left\langle \hat{S}_y \right\rangle$ and $\left\langle \hat{S}_z \right\rangle$ depend on time for $t > 0$.

Hint: for short notation you could use the results $\left\langle \uparrow \middle| \hat{S}_j \middle| \uparrow \right\rangle$ etc. of question d).

f)

As follow up on question e), make graphs for $\left\langle \hat{S}_x \right\rangle$, $\left\langle \hat{S}_y \right\rangle$ and $\left\langle \hat{S}_z \right\rangle$ as a function of time for $t \geq 0$ (so, you must make 3 graphs). Describe in words what the dynamics is of the spins in question e). If you like you can make a drawing with your explanation.

g)

If you want to measure the time-dependent spin dynamics of such spins (for example the dynamics of question e) and f)), how could you measure that? Explain in some detail (10 lines of text) what the physical signal is that you would try to detect, and what apparatus you could build for that.

Problem W6.2

(Problem taken from exam 1-11-2012, now added as homework problem per 2014)

a) The Hamiltonian is only a function of $\hat{S}_z$, so $\hat{H}$ and $\hat{S}_z$ commute ($[\hat{H}, \hat{S}_z] = 0$).

$\hat{S}_x$ and $\hat{S}_y$ do not commute with the Hamiltonian because $[\hat{S}_x, \hat{S}_z] \neq 0$ and $[\hat{S}_y, \hat{S}_z] \neq 0$.

b) The Hamiltonian has the same eigenstates as $\hat{S}_z$, so these are simply

$$\begin{cases} |\uparrow\rangle & \text{(spin-up along the z-axis)} \\ |\downarrow\rangle & \text{(spin-down along the z-axis)} \end{cases}$$

The energy eigenvalue for the state $|\uparrow\rangle$ is

$$\hat{H}|\uparrow\rangle = \underbrace{-\gamma B_z \cdot +\tfrac{1}{2}\hbar}_{\text{energy eigenvalue for } |\uparrow\rangle} |\uparrow\rangle \quad \text{, where we use } \hat{S}_z|\uparrow\rangle = \tfrac{1}{2}\hbar|\uparrow\rangle$$

$$\Rightarrow \boxed{E_\uparrow = -\tfrac{1}{2}\gamma\hbar B_z}$$

The energy eigenvalue for the state $|\downarrow\rangle$ is

$$H|\downarrow\rangle = \underbrace{-\gamma B_z \cdot -\tfrac{1}{2}\hbar}|\downarrow\rangle, \quad \text{where we used } \hat{S}_z|\downarrow\rangle = -\tfrac{1}{2}\hbar|\downarrow\rangle$$

$\Rightarrow$ the energy eigenvalue for $|\downarrow\rangle$ is

$$\boxed{E_\downarrow = +\tfrac{1}{2}\gamma\hbar B_z}$$

$E_\uparrow < E_\downarrow$, so $|\uparrow\rangle$ is the ground state and $|\downarrow\rangle$ the only excited state.

c) Resonant driving of quantum dynamics requires

$$hf = E_\downarrow - E_\uparrow = \gamma \hbar B_z \Rightarrow \boxed{f = \frac{\gamma B_z}{2\pi}} \Rightarrow$$

$$f = 85.15 \text{ MHz}$$

d) $\boxed{\hat{S}_z}$

$$\langle \uparrow | \hat{S}_z | \uparrow \rangle = +\tfrac{1}{2}\hbar$$
$$\langle \downarrow | \hat{S}_z | \downarrow \rangle = -\tfrac{1}{2}\hbar$$
$$\langle \uparrow | \hat{S}_z | \downarrow \rangle = 0$$
$$\langle \downarrow | \hat{S}_z | \uparrow \rangle = 0$$

Further use the matrix representation in S_z basis

$$|\uparrow\rangle \Leftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\downarrow\rangle \Leftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\hat{S}_x \Leftrightarrow \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{S}_y \Leftrightarrow \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$\boxed{S_x}$

$$\langle \uparrow | \hat{S}_x | \uparrow \rangle = \frac{\hbar}{2}(1\ 0)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$$

$$\langle \downarrow | \hat{S}_x | \downarrow \rangle = \frac{\hbar}{2}(0\ 1)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

$$\langle \uparrow | \hat{S}_x | \downarrow \rangle = \frac{\hbar}{2}(1\ 0)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = +\frac{\hbar}{2}$$

$$\langle \downarrow | \hat{S}_x | \uparrow \rangle = \frac{\hbar}{2}(0\ 1)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = +\frac{\hbar}{2}$$

$\boxed{S_y}$

$$\langle \uparrow | \hat{S}_y | \uparrow \rangle = \frac{\hbar}{2}(1\ 0)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$$

$$\langle \downarrow | \hat{S}_y | \downarrow \rangle = \frac{\hbar}{2}(0\ 1)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

$$\langle \uparrow | \hat{S}_y | \downarrow \rangle = \frac{\hbar}{2}(1\ 0)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -i\frac{\hbar}{2}$$

$$\langle \downarrow | \hat{S}_y | \uparrow \rangle = \frac{\hbar}{2}(0\ 1)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = +i\frac{\hbar}{2}$$

e) Define $\omega_\uparrow = \frac{E_\uparrow}{\hbar}$ and $\omega_\downarrow = \frac{E_\downarrow}{\hbar}$

$$|\psi(t)\rangle = e^{-i/\hbar \, \hat{H} t}|\psi_0\rangle = \sqrt{\tfrac{2}{3}}\, e^{-i\omega_\uparrow t}|\uparrow\rangle + i\sqrt{\tfrac{1}{3}}\, e^{-i\omega_\downarrow t}|\downarrow\rangle$$

$$\langle \hat{S}_z\rangle(t) = \langle \psi(t)|\hat{S}_z|\psi(t)\rangle$$

$$= \left(\sqrt{\tfrac{2}{3}}\, e^{+i\omega_\uparrow t}\langle\uparrow| - i\sqrt{\tfrac{1}{3}}\, e^{+i\omega_\downarrow t}\langle\downarrow|\right)\hat{S}_z\left(\sqrt{\tfrac{2}{3}}\, e^{-i\omega_\uparrow t}|\uparrow\rangle + i\sqrt{\tfrac{1}{3}}\, e^{-i\omega_\downarrow t}|\downarrow\rangle\right)$$

$$= \tfrac{2}{3}\langle\uparrow|S_z|\uparrow\rangle + \tfrac{1}{3}\langle\downarrow|\hat{S}_z|\downarrow\rangle$$

$$= +\tfrac{2}{3}\cdot\tfrac{1}{2}\hbar - \tfrac{1}{3}\cdot\tfrac{1}{2}\hbar = \tfrac{1}{6}\hbar$$

$$\langle \hat{S}_x\rangle(t) = \langle \psi(t)|\hat{S}_x|\psi(t)\rangle$$

$$= \left(\sqrt{\tfrac{2}{3}}\, e^{+i\omega_\uparrow t}\langle\uparrow| - i\sqrt{\tfrac{1}{3}}\, e^{+i\omega_\downarrow t}\langle\downarrow|\right)\hat{S}_x\left(\sqrt{\tfrac{2}{3}}\, e^{-i\omega_\uparrow t}|\uparrow\rangle + i\sqrt{\tfrac{1}{3}}\, e^{-i\omega_\downarrow t}|\downarrow\rangle\right)$$

$$= \frac{\sqrt{2}}{3}e^{-i(\omega_\downarrow-\omega_\uparrow)t}\cdot i\,\langle\uparrow|\hat{S}_x|\downarrow\rangle - \frac{\sqrt{2}}{3}e^{+i(\omega_\downarrow-\omega_\uparrow)t}\cdot i\,\langle\downarrow|\hat{S}_x|\uparrow\rangle$$

$$= \frac{\sqrt{2}}{3}\, i\, \frac{\hbar}{2}\left(e^{-i(\omega_\downarrow-\omega_\uparrow)t} - e^{+i(\omega_\downarrow-\omega_\uparrow)t}\right)$$

$$= (\cancel{\cos} - i\sin) - (\cancel{\cos} + i\sin)$$

$$= \frac{\sqrt{2}\,\hbar}{6}\, i\left(-i\sin(\omega_\downarrow-\omega_\uparrow)t - i\sin(\omega_\downarrow-\omega_\uparrow)t\right)$$

$$= \frac{\sqrt{2}}{3}\hbar\,\sin(\omega_\downarrow-\omega_\uparrow)t$$

$$\langle S_y\rangle(t) = \langle \psi(t)|\,\hat{S}_y\,|\psi(t)\rangle$$

$$= \left(\sqrt{\tfrac{2}{3}}\, e^{+i\omega_\uparrow t}\langle\uparrow| - i\sqrt{\tfrac{1}{3}}\, e^{+i\omega_\downarrow t}\langle\downarrow|\right)\hat{S}_y\left(\sqrt{\tfrac{2}{3}}\, e^{-i\omega_\uparrow t}|\uparrow\rangle + i\sqrt{\tfrac{1}{3}}\, e^{-i\omega_\downarrow t}|\downarrow\rangle\right)$$

$$= \frac{\sqrt{2}}{3} e^{-i(\omega_\downarrow-\omega_\uparrow)t}\cdot i\,\underbrace{\langle\uparrow|\hat{S}_y|\downarrow\rangle}_{-i\frac{\hbar}{2}} - \frac{\sqrt{2}}{3} e^{+i(\omega_\downarrow-\omega_\uparrow)t}\cdot i\,\underbrace{\langle\downarrow|\hat{S}_y|\uparrow\rangle}_{+i\frac{\hbar}{2}}$$

$$= \frac{\sqrt{2}}{3} e^{-i(\omega_\downarrow-\omega_\uparrow)t}\cdot\frac{\hbar}{2} + \frac{\sqrt{2}}{3} e^{+i(\omega_\downarrow-\omega_\uparrow)t}\cdot\frac{\hbar}{2}$$

$$= \frac{\sqrt{2}}{3}\hbar\cos(\omega_\downarrow-\omega_\uparrow)t$$

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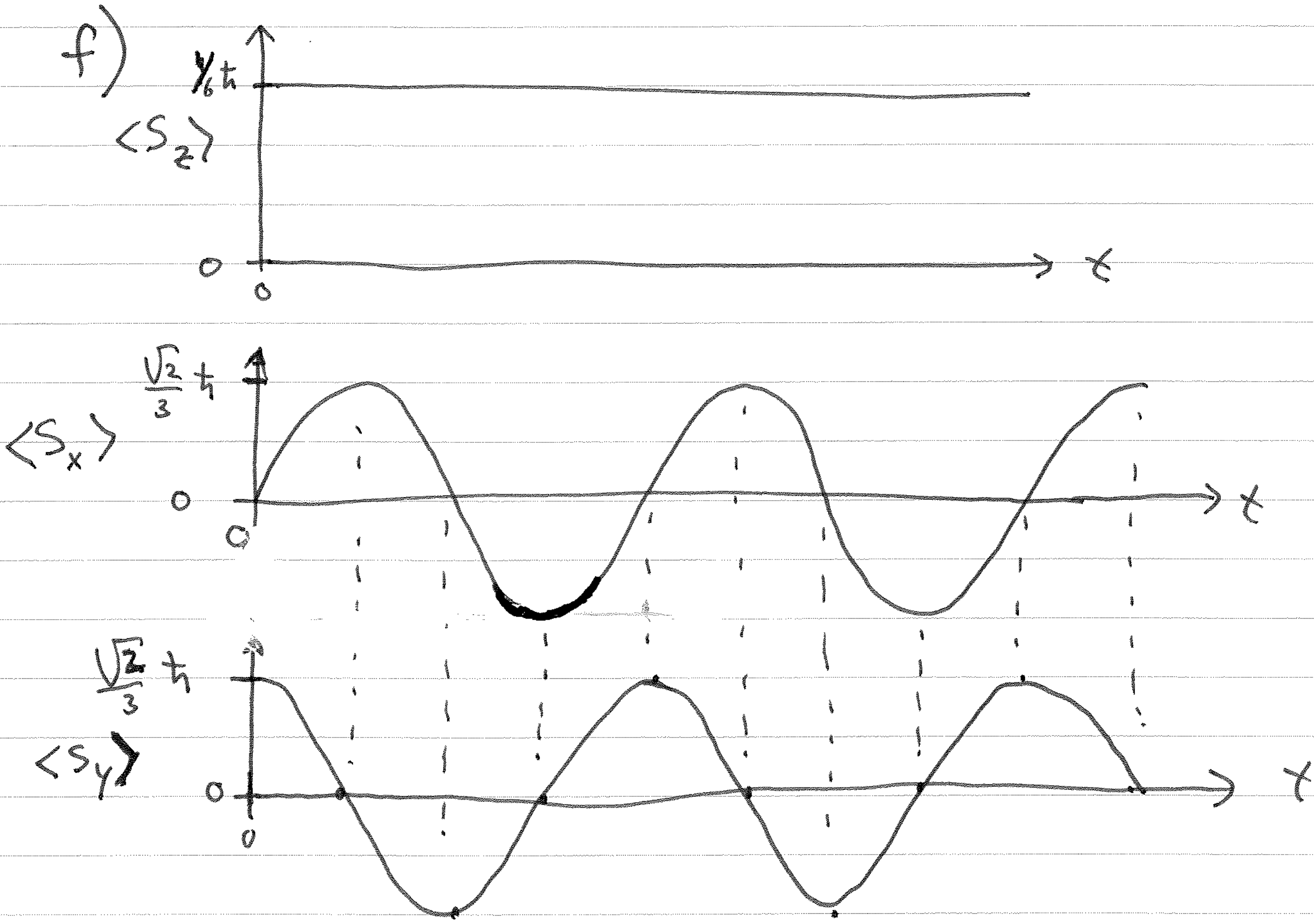
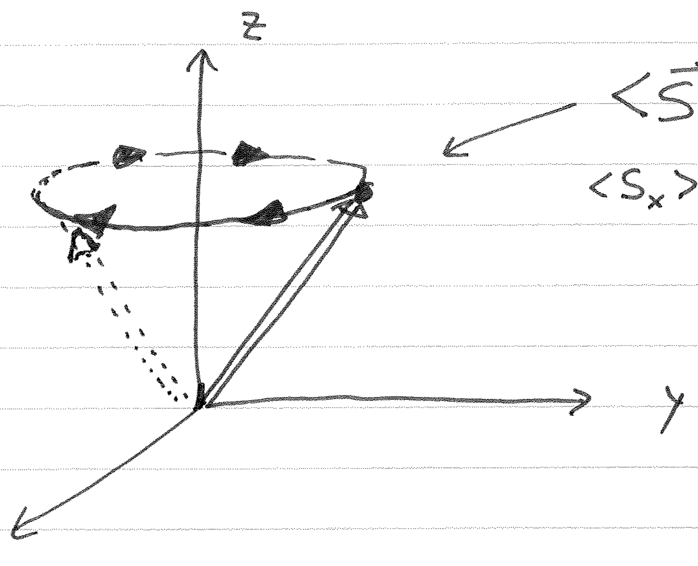
f)



The spin is precessing around the field $\vec{B} = B_z \cdot \vec{e}_z$ unit vector in z-direction



$\langle \vec{S} \rangle = \langle S_x \rangle \vec{e}_x + \langle S_y \rangle \vec{e}_y + \langle S_z \rangle \vec{e}_z$

See also Griffiths book p. 181

g) Each spin has a magnetic moment $\vec{\mu} = \gamma \vec{S}$. So if the spins precess all together there is a rotating macroscopic magnetic moment (magnetization). You can measure that with a coil (electrical wire in loops). The oscillating magnetization induces a current (or electromagnetic force e.m.f) in the coil that you can detect (like the dynamo on your bike for getting electrical current in your bike lights).