Problem set for 3rd week of the course Quantum Physics 1

(version 2015-2016, still valid in later years if handed out as a print or provided on the course website)

Homework, to be made before the werkcollege:
From the book (Griffiths 2nd Ed.) Chapter 2 - 2.18, 2.19 (just use the result stated in problem 1.14. if you did not yet make that problem), 2.21 and from Chapter 3 - 3.1, 3.3, 3.22.

Problems to work on during werkcollege:
Problems W3.1 – W3.5 (this hand out) and from the book Chapter 3 - 3.5, this is the minimal set you need to do. Other good problems that we selected (we advise you to make these for the topics where you need or like to do extra training):
from the book Chapter 2 - 2.20, and from Chapter 3 - 3.2, 3.4, 3.21, 3.23 (work this problem also out by using a matrix representation that uses the vectors $|1\rangle$ and $|2\rangle$ as basis vectors), 3.24.

NOTE: Fourier transform relations between *x*- and *k*-representation of a state:

$$\overline{\Psi}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx}\ \Psi(x)\, dx$$

$$\Psi(x) \ = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx}\ \overline{\Psi}(k) dk$$

Problem W3.1

This problem (about a particle in a box) is meant to clarify how the <u>same</u> *state* of quantum system can be *represented* in many different ways. The representations and notations that will be used here are:
- describing a state using Dirac notation.
- a wavefunction that is a function of position x.
- a wavefunction that is a function of wave number k (or, equivalently, momentum $p_x=\hbar k$).
- a superposition of energy eigenstates.

On the way, you will practice with Fourier transforms and decomposition of a state into eigenvectors.

a) The particle in the box is modeled as a particle in an infinitely deep potential well with V=0 for $|x|<a/2$, and $V=\infty$ elsewhere. The particle is brought into the box with a mechanism that results in a wavefunction for the particle that is evenly distributed in the well, $\Psi(x)=1/\sqrt{a}$ for $|x|<a/2$ and zero elsewhere. Represent this state in the *k*-representation (hint: you need to Fourier transform the state).

b) Alternatively, this state can for example be represented as a superposition of energy eigenstates of the system in Dirac notation, $|\Psi\rangle = \sum_n c_n |\varphi_n\rangle$. We will use this later in this problem. In this question we first pay attention to the relation between the *x*-representation and the representation with Dirac notation. Prove the relation $\Psi(x) = \langle x|\Psi\rangle$ (here $|x\rangle$ is the eigenvector with eigenvalue x for the position operator $\hat{x}$, which means that the state $|x\rangle$ in Dirac notation corresponds to $\delta(x'-x)$ in *x*-representation, see also Griffiths Eqs. [2.111]-[2.113] and [3.34]-[3.42]).

c) Eq. [2.28] of the Griffiths book ($\psi_n(x)$ in Sec. 2.2) gives the energy eigenstates for a particle in this system, but for a box (or quantum well) that runs from x=0 to $x=a$. Here we use a description where the box runs from $x=-a/2$ to $x=a/2$ (and slightly different notation with $\varphi_n(x)$ for the energy eigenstates, instead of $\psi_n(x)$ as in the book). The energy eigenvalues E_n are of course the same, but the eigenstates are now

$$\varphi_n(x) = \sqrt{\tfrac{2}{a}}\cos\left(\tfrac{n\pi x}{a}\right), \qquad \text{for } n = 1, 3, 5, 7, \ldots$$

$$\varphi_n(x) = \sqrt{\tfrac{2}{a}}\sin\left(\tfrac{n\pi x}{a}\right), \qquad \text{for } n = 2, 4, 6, 8, \ldots$$

Write down the eigenfunctions for odd n in the *k*-representation (you need to use the Fourier transform, see also Griffiths Eqs. [2.101]-[2.103] and [3.54]-[3.55]).

d) Sketch the wavefunction of the particle (for the state as in question **a)**), as well as the energy eigenstates $|\varphi_1\rangle$ and $|\varphi_9\rangle$ in the k-representation. Explain the differences between the graphs.
Hint 1: you need to sketch here a *sinc function* or the sum of two shifted *sinc functions*. In its most basic form the sinc function is sinc(x)=sin(x)/x. It is easy to construct as follows: Sketch sin(x), sketch 1/x, and multiply the two graphs. Also look up the value of the limit of sin(x)/x for $x \to 0$ (see for example Griffiths Eqs. [2.103]-[2.105]).
Hint 2: for $\Psi(x)$ in k-representation write it in the form of a sinc function. For $|\varphi_1\rangle$ and $|\varphi_9\rangle$ in k-representation, try to write this as the sum of two shifted sinc functions. That is, for n = 1 and n = 9 your answer should have a term that contains the factor $\mathrm{sinc}((k-\frac{n\pi}{a})\frac{a}{2})$ and a term that contains the factor $\mathrm{sinc}((k+\frac{n\pi}{a})\frac{a}{2})$.

e) For continuing on **b)**, you need to determine the coefficients c_n for odd n. Prove the relation $c_n=\langle\varphi_n|\Psi\rangle$ in Dirac notation (this quantity c_n is often called the projection of $|\Psi\rangle$ onto $|\varphi_n\rangle$).

f) Evaluate the inner product $c_n=\langle\varphi_n|\Psi\rangle$ for odd n in the x-representation.

g) Write down the inner product $c_n=\langle\varphi_n|\Psi\rangle$ for odd n in the k-representation, but only solve the integral if you feel like doing so.

h) Without doing the calculation, can you say what the value is of $c_n=\langle\varphi_n|\Psi\rangle$ for even n.

Problem W3.2
In this problem use the definitions that for an *even* function $f(x)=f(-x)$ and that for an *odd* function $f(x)=-f(-x)$. Say $\varphi(x)$ is an arbitrary function, and we introduce

$$\varphi_e(x)=\frac{\varphi(x)+\varphi(-x)}{2}$$
$$\varphi_o(x)=\frac{\varphi(x)-\varphi(-x)}{2}$$

Prove that $\varphi_e(x)$ is an even function, that $\varphi_o(x)$ is an odd function, and express $\varphi(x)$ in terms of $\varphi_e(x)$ and $\varphi_o(x)$. Remember these properties for the rest of your life.

Problem W3.3
Consider an electron, that behaves as a one-dimensional quantum particle.

a) At some time t_0 the electron is in the state $\Psi(x) = Ae^{\frac{-|x|}{a}}$, where a and A real and positive. For which value of A is this state normalized?

b) Derive an expression that describes this state as a superposition of plane waves with wavenumber k.

c) *Roughly estimate* Δx and Δp_x for the state of question **a)**, and check whether it violates the Heisenberg uncertainty relation.

d) One wants to measure the velocity of this particle at time t_0. Calculate the probability for getting a result between 40 km/s and 50 km/s (calculate it here for the case of phase velocity rather than group velocity). Calculate a numerical result, use a = 1 nm. You may need to use this solution to the following integral.

$$\int\left(\frac{1}{1+b^2y^2}\right)^2 dy=\frac{1}{2}\left(\frac{y}{1+b^2y^2}+\frac{\arctan(by)}{b}\right).$$

Problem W3.4
A single electron is in a large empty space, where no fields or external forces act on this "free particle". We will only look at its behavior along the x-direction. At some time $t = t_o$, the wavefunction that describes the linear momentum p_x of the electron is (here a is real and positive)

$$\Psi_p(p_x, t_0) = Ce^{-(p_x/a)^2} \quad .$$

a) If we would now measure the electron's velocity, what value are we most likely going to find?

b) We decided not to measure the velocity, because we did not know well enough yet where the electron was. What was at t_o approximately the lower limit in the fundamental uncertainty in the x-position of the electron, expressed in meters? Use $a = 9.109 \cdot 10^{-31}$ kg m s^{-1}.

c) We are happy with the answer on **b)**, the electron is well enough localized in the volume where we can do the experiments. However, our thinking about it took 100 seconds. Estimate how much the lower limit in the uncertainty in the x-position of the electron increased during this time.

d) We decide to measure now as soon as possible. What is now the value for the electron's velocity that we are most likely going to find?

Problem W3.5
When working with states and operators in formulas and equations it is important to write them down in the proper order, and to take the complex conjugates and hermitian adjoints of operators (sometimes called hermitian conjugate) in the proper way when needed. To practice with this, we ask you in this problem to write what is presented in words as a formula. Where possible you must use Dirac notation. In your final answer, the operators must be outside the brackets $|\,\rangle$ and $\langle\,|$ of the bra and ket notation.
Note: if your answer differs from the answer sheets on any detail, you possibly have it really wrong.

a) Operator $\hat{A}$ works on the ket-vector representing the state Ψ.

b) Operator $\hat{A}$ works on the bra-vector representing the state Ψ.

c) Operator $\hat{A}$ works on the ket-vector representing the state Ψ, and then operator $\hat{A}$ works on the result of this one more time.

d) Operator $\hat{A}$ works on the ket-vector representing the state Ψ, and then operator $\hat{B}$ works on the result of this.

e) Operator $\hat{A}$ works on the bra-vector representing the state Ψ, and then operator $\hat{B}$ works on the result of this.

f) Operator $e^{i\varphi}\hat{A}$ (with phase factor) works on the ket-vector representing the state Ψ.

g) Operator $e^{i\varphi}\hat{A}$ (with phase factor) works on the bra-vector representing the state Ψ.

h) Operator $\hat{A}$ works on the ket-vector representing the state that is a superposition of the states φ_1 and φ_2 with the complex probabilities amplitudes α and β, respectively.

i) Operator $\hat{A}$ works on the bra-vector representing the state that is a superposition of the states φ_1 and φ_2 with the complex probabilities amplitudes α and β, respectively.

j) A point that may need clarification is how operators like the time evolution operator $\hat{U} = e^{-\frac{i\hat{H}t}{\hbar}}$ that have an operator in the exponent work on a state. Write out a Taylor expansion around $t = 0$ for this operator to show that the way that it works on the ket-vector representing the state Ψ gives terms of the form that already appeared for the answer on **a)** and **c)**.