Quantum Physics 1

Some notes on the topics angular momentum and spin

Here not the derivation of equations or the physics,
but a summary of rules and various notations

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From classical mechanics, basics you should already be able to dream..

$$\vec{L} = \vec{r} \times \vec{p}$$

$$\vec{L} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix} = \vec{e}_x\left(yp_z - zp_y\right) + \vec{e}_y\left(zp_x - xp_z\right) + \vec{e}_z\left(xp_y - yp_x\right)$$

$$= L_x\,\vec{e}_x + L_y\,\vec{e}_y + L_z\,\vec{e}_z$$

where the $\vec{e}_x$ etc. are unit vectors,

and where L_x, L_y , L_z are scalars (not vectors).

$$\left(\vec{L}\right)^2 = \vec{L} \cdot \vec{L} = L_x^2 + L_y^2 + L_z^2$$

is also a scalar (not a vector) which characterizes the length of $\vec{L}$.

The previous slide can be used to show that

for angular momentum as an operator $\hat{L}$

$$\left[\hat{L}_x, \hat{L}_y\right] = i\hbar\hat{L}_z$$
$$\left[\hat{L}_z, \hat{L}_x\right] = i\hbar\hat{L}_y$$
$$\left[\hat{L}_y, \hat{L}_z\right] = i\hbar\hat{L}_x$$
$$\left[\hat{L}^2, \hat{L}_x\right] = \left[\hat{L}^2, \hat{L}_y\right] = \left[\hat{L}^2, \hat{L}_z\right] = 0$$

You can derive this with the previous slide, while filling in

$\hat{x}, \hat{y}, \hat{z},$ and $\hat{p}_x = -i\hbar \dfrac{\partial}{\partial x}$ etc.

and using

$$\left[\hat{x}, \hat{p}_x\right] = i\hbar \quad etc., \quad \left[\hat{x}, \hat{y}\right] = 0 \quad etc., \quad \left[\hat{x}, \hat{p}_y\right] = 0 \quad etc., \quad \left[\hat{p}_x, \hat{p}_y\right] = 0$$

etc.

In turn, the commutation relations on the previous slide can be used to derive that the eigenvalue equations for angular momentum are ALWAYS in the following form (sec. 4.3 Griffiths book), with l and m integer or half-integer numbers, as further introduced on the remaining slides

$$\hat{L}_z \; |l,m\rangle = \hbar m \; |l,m\rangle$$

$$\hat{L}^2 \; |l,m\rangle = \hbar^2 \; l(l+1) \; |l,m\rangle$$

Many different symbols used for angular momentum

Symbol	Often used for
$\vec{L}$	The orbital angular momentum of an electron in an atom
$\vec{S}$	The spin angular momentum of an electron
$\vec{J} = \vec{L} + \vec{S}$	The sum of spin and orbital angular momentum of an electron in an atom
$\vec{I}$	The spin angular momentum of a nucleus
$\vec{F} = \vec{J} + \vec{I}$	The total angular momentum of an atom (with one electron) from the nucleus and the electron
$\vec{L} = \vec{L}_1 + \vec{L}_2$	The total orbital angular momentum of an atom with two electrons
$\vec{S} = \vec{S}_1 + \vec{S}_2$	The total electronic spin angular momentum of an atom with two electrons

Eigenvalue equations for angular momentum

for $\vec{L}, \vec{S}, \vec{J}$ etc. ALWAYS have the same structure

Here worked out for $\vec{J}$.

Eigenvalue equation for **length of the angular momentum vector**, also called **total angular momentum**:

$$\hat{J}^2 \left| j, m_{jz} \right\rangle = \hbar^2 \; j(j+1) \left| j, m_{jz} \right\rangle$$

where the length is then in fact $\hbar\sqrt{j(j+1)}$.

j is the quantum number for the length of the angular momentum vector.

The widely used convention is to work with $\hat{J}^2$,
but you could also introduce an operator that directly gives you the length (see next slide).

Note: For the notation of the ket states $\left| j, m_{jz} \right\rangle$ you may also see $\left| j \; m_{jz} \right\rangle$ or $\left| \varphi_{j,m_{jz}} \right\rangle$ etc.

Eigenvalue equations for angular momentum - continued

Eigenvalue equation for **length of the angular momentum vector**, also called **total angular momentum**.

The widely used convention is to work with $\hat{J}^2$,
but you could also introduce an operator that directly gives you the length.

If you define that the operator $\widehat{|J|}$ represents the length of $\vec{J}$,
then its eigenvalue equation obeys

$$\widehat{|J|} \; \left| j, m_{jz} \right\rangle = \hbar \sqrt{j(j+1)} \; \left| j, m_{jz} \right\rangle$$

Eigenvalue equations for angular momentum - continued

Eigenvalue equations for the **x, y and z component of the angular momentum vector**:

$$\hat{J}_z \left| j, m_{jz} \right\rangle = \hbar m_{jz} \left| j, m_{jz} \right\rangle$$

m_{jz} is the quantum number for the z component of the angular momentum vector (sometimes called the z magnetic quantum number).

In the same way, for the x and y components:

$$\hat{J}_x \left| j, m_{jx} \right\rangle = \hbar m_{jx} \left| j, m_{jx} \right\rangle$$

$$\hat{J}_y \left| j, m_{jy} \right\rangle = \hbar m_{jy} \left| j, m_{jy} \right\rangle$$

Eigenvalue equations for angular momentum - continued

The quantum numbers only appear with these values:

If $\vec{J}$ represents an orbital angular momentum j can take on these values

$$j = 0, 1, 2, 3, \ldots$$

If $\vec{J}$ represents the intrinsic angular momentum of a particle (called spin) j can appear with these values

$$j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \tfrac{5}{2}, \ \ldots$$

For both cases, given a certain value for j, the values for m_{jz} must obey

$$m_{jz} = -j, -(j-1), -(j-2), \ \ldots \ +(j-1), +j$$

and similarly for m_{jx} and m_{jy}

$$m_{jx} = -j, -(j-1), -(j-2), \ \ldots \ +(j-1), +j$$

$$m_{jy} = -j, -(j-1), -(j-2), \ \ldots \ +(j-1), +j$$

Symbols used for the quantum numbers

Angular momentum vector	$\vec{L}$	$\vec{S}$	$\vec{J}$	$\vec{I}$	$\vec{F}$	$\vec{S}_1$
Quantum number for length	l	s	j	i	f	s_1
Quantum number for x component	m_{lx}	m_{sx}	m_{jx}	m_{ix}	m_{fx}	m_{s1x}
Quantum number for y component	m_{ly}	m_{sy}	m_{jy}	m_{iy}	m_{fy}	m_{s1y}
Quantum number for z component	m_{lz}	m_{sz}	m_{jz}	m_{iz}	m_{fz}	m_{s1z}

Note: for the m quantum numbers the subscript indices are often not used in cases where leaving them away will not give confusion.
For example, instead of m_{jz} you will often simply see m, m_j or m_z .

Summary of spin-1/2 operators and eigenstates

(using Griffith book Eqs. [4.136], [4.139]-[4.152])

$$\hat{S}_z\left|\uparrow_z\right\rangle = +\tfrac{1}{2}\hbar\left|\uparrow_z\right\rangle$$

$$\hat{S}_z\left|\downarrow_z\right\rangle = -\tfrac{1}{2}\hbar\left|\downarrow_z\right\rangle$$

$$\left|\uparrow_z\right\rangle \leftrightarrow \begin{pmatrix}1\\0\end{pmatrix} \quad \text{and} \quad \left|\downarrow_z\right\rangle \leftrightarrow \begin{pmatrix}0\\1\end{pmatrix} \text{ gives } \quad \hat{S}_z \leftrightarrow \frac{\hbar}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$

$$\hat{S}_x \leftrightarrow \frac{\hbar}{2}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \qquad \hat{S}_y \leftrightarrow \frac{\hbar}{2}\begin{pmatrix}0 & -i\\i & 0\end{pmatrix}$$

$$\begin{cases}\left|\uparrow_x\right\rangle = \dfrac{1}{\sqrt{2}}\left(\left|\uparrow_z\right\rangle + \left|\downarrow_z\right\rangle\right)\\ \left|\downarrow_x\right\rangle = \dfrac{1}{\sqrt{2}}\left(\left|\uparrow_z\right\rangle - \left|\downarrow_z\right\rangle\right)\end{cases} \Leftrightarrow \begin{cases}\left|\uparrow_z\right\rangle = \dfrac{1}{\sqrt{2}}\left(\left|\uparrow_x\right\rangle + \left|\downarrow_x\right\rangle\right)\\ \left|\downarrow_z\right\rangle = \dfrac{1}{\sqrt{2}}\left(\left|\uparrow_x\right\rangle - \left|\downarrow_x\right\rangle\right)\end{cases}$$

$$\begin{cases}\left|\uparrow_y\right\rangle = \dfrac{1}{\sqrt{2}}\left(\left|\uparrow_z\right\rangle + i\left|\downarrow_z\right\rangle\right)\\ \left|\downarrow_y\right\rangle = \dfrac{1}{\sqrt{2}}\left(\left|\uparrow_z\right\rangle - i\left|\downarrow_z\right\rangle\right)\end{cases} \Leftrightarrow \begin{cases}\left|\uparrow_z\right\rangle = \dfrac{1}{\sqrt{2}}\left(\left|\uparrow_y\right\rangle + \left|\downarrow_y\right\rangle\right)\\ \left|\downarrow_z\right\rangle = \dfrac{1}{i\sqrt{2}}\left(\left|\uparrow_y\right\rangle - \left|\downarrow_y\right\rangle\right)\end{cases}$$

Addition of angular momentum

Say, we consider the addition $\vec{J} = \vec{L} + \vec{S}$, in a system that combines two parts, where one part has angular momentum $\vec{L}$ and the other part angular momentum $\vec{S}$.

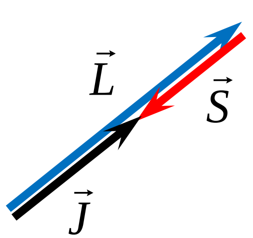
Is the physics that you want/need to describe a function of $\vec{J}$ or $\vec{L}$ or $\vec{S}$?

If you do a measurement of angular momentum on this system, do you measure $\vec{J}$ or $\vec{L}$ or $\vec{S}$?

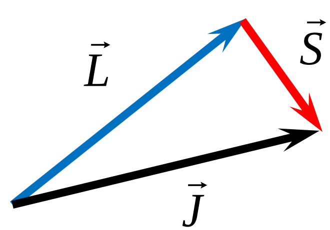
In case that the answer is $\vec{J}$, what is its behavior?

For intuitively understanding the next slide (range of j and m_j quantum numbers), consider these cases, with for each case the same lengths of $\vec{L}$ and $\vec{S}$.

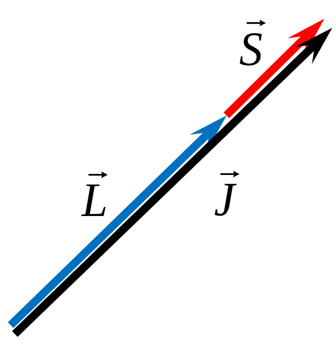
Anti-parallel



Non-parallel



Parallel



Addition of angular momentum

Say, we consider the addition $\vec{J} = \vec{L} + \vec{S}$.

What are now the eigenvalues and eigenstates of $\vec{J}$?

1) As ALWAYS, the eigenvalues and states of $\vec{J}$ must obey the usual structure:

$$\hat{J}^2 \left| j, m_{jz} \right\rangle = \hbar^2 \; j(j+1) \left| j, m_{jz} \right\rangle$$

$$\hat{J}_z \left| j, m_{jz} \right\rangle = \hbar \, m_{jz} \left| j, m_{jz} \right\rangle$$

2) What are the possible quantum numbers j, given certain values for l and s ?

$$j = |l-s|, \;\; |l-s|+1, \;\; |l-s|+2, ... \;\; (l+s-2), \;\; (l+s-1), \;\; (l+s)$$

3) What are the possible quantum numbers for m_{jz}, given a certain value for j?

$$m_{jz} = -j, -(j-1), -(j-2), \;\; ... \;\; +(j-1), + j$$

Note that only 2) is something new. Also note that the range of j values can be seen as ranging from $\vec{L}$ and $\vec{S}$ being fully parallel to fully anti-parallel.

NOTE!

The following (all remaining) slides are added at the request of the students after we used them in a lecture.

They may be handy will working on problems or checking theory, but don't read them as if they are core study material.

Expectation values and uncertainties for x-, y-, and z-components of spin for a spin-½ system in the spin-up or spin-down state

First we list the states and operators expressed as vectors and matrices in a representation that uses the spin-up state $|\uparrow_z\rangle$ and spin-down state $|\downarrow_z\rangle$ along the z-direction as basis states. We also define expression for the quantum uncertainties.

$$|\uparrow_z\rangle \leftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|\downarrow_z\rangle \leftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\hat{S}_x \leftrightarrow \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\hat{S}_y \leftrightarrow \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\hat{S}_z \leftrightarrow \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\hat{S}_x^2 \leftrightarrow \frac{1}{4}\hbar^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\hat{S}_y^2 \leftrightarrow \frac{1}{4}\hbar^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\hat{S}_z^2 \leftrightarrow \frac{1}{4}\hbar^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2 \leftrightarrow \frac{3}{4}\hbar^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Delta S_x = \sqrt{\langle \hat{S}_x^2 \rangle - \langle \hat{S}_x \rangle^2}$$

$$\Delta S_y = \sqrt{\langle \hat{S}_y^2 \rangle - \langle \hat{S}_y \rangle^2}$$

$$\Delta S_z = \sqrt{\langle \hat{S}_z^2 \rangle - \langle \hat{S}_z \rangle^2}$$

The entries to the table (next slide) are calculated as follows.

$$\langle \uparrow_z | \hat{S}_x | \uparrow_z \rangle = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0, \quad \text{etc.}$$

Values for $\left|\uparrow_z\right\rangle \leftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\left\langle \uparrow_z \right| \hat{S}_x \left| \uparrow_z \right\rangle = 0$$

$$\left\langle \uparrow_z \right| \hat{S}_y \left| \uparrow_z \right\rangle = 0$$

$$\left\langle \uparrow_z \right| \hat{S}_z \left| \uparrow_z \right\rangle = +\frac{\hbar}{2}$$

$$\left\langle \uparrow_z \right| \hat{S}_x^2 \left| \uparrow_z \right\rangle = \tfrac{1}{4}\hbar^2$$

$$\left\langle \uparrow_z \right| \hat{S}_y^2 \left| \uparrow_z \right\rangle = \tfrac{1}{4}\hbar^2$$

$$\left\langle \uparrow_z \right| \hat{S}_z^2 \left| \uparrow_z \right\rangle = \tfrac{1}{4}\hbar^2$$

$$\left\langle \uparrow_z \right| \hat{S}^2 \left| \uparrow_z \right\rangle = \tfrac{3}{4}\hbar^2$$

$$\Delta S_x = \frac{\hbar}{2}$$

$$\Delta S_y = \frac{\hbar}{2}$$

$$\Delta S_z = 0$$

Values for $\left|\downarrow_z\right\rangle \leftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\left\langle \downarrow_z \right| \hat{S}_x \left| \downarrow_z \right\rangle = 0$$

$$\left\langle \downarrow_z \right| \hat{S}_y \left| \downarrow_z \right\rangle = 0$$

$$\left\langle \downarrow_z \right| \hat{S}_z \left| \downarrow_z \right\rangle = -\frac{\hbar}{2}$$

$$\left\langle \downarrow_z \right| \hat{S}_x^2 \left| \downarrow_z \right\rangle = \tfrac{1}{4}\hbar^2$$

$$\left\langle \downarrow_z \right| \hat{S}_y^2 \left| \downarrow_z \right\rangle = \tfrac{1}{4}\hbar^2$$

$$\left\langle \downarrow_z \right| \hat{S}_z^2 \left| \downarrow_z \right\rangle = \tfrac{1}{4}\hbar^2$$

$$\left\langle \downarrow_z \right| \hat{S}^2 \left| \downarrow_z \right\rangle = \tfrac{3}{4}\hbar^2$$

$$\Delta S_x = \frac{\hbar}{2}$$

$$\Delta S_y = \frac{\hbar}{2}$$

$$\Delta S_z = 0$$

Matrices in z-basis for angular momentum in x-, y-, and z-direction
for j = ½ and j = 1

$\hat{J}_z \; for \; j=\frac{1}{2}$

$$\hat{J}_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$m_z = +\frac{1}{2}$ $$\frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\cdot\begin{pmatrix}1\\ 0\end{pmatrix} = +\frac{1}{2}\hbar\cdot\begin{pmatrix}1\\ 0\end{pmatrix}$$

$m_z = -\frac{1}{2}$ $$\frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\cdot\begin{pmatrix}0\\ 1\end{pmatrix} = -\frac{1}{2}\hbar\cdot\begin{pmatrix}0\\ 1\end{pmatrix}$$

$\hat{J}_x \; for \; j=\frac{1}{2}$

$$\hat{J}_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

$m_x = +\frac{1}{2}$ $$\frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\cdot\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix} = +\frac{1}{2}\hbar\cdot\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$$

$m_x = -\frac{1}{2}$ $$\frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\cdot\begin{pmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{pmatrix} = -\frac{1}{2}\hbar\cdot\begin{pmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{pmatrix}$$

$\hat{J}_y \; for \; j=\frac{1}{2}$

$$\hat{J}_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$$

$m_y = +\frac{1}{2}$ $$\frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\cdot\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{i}{\sqrt{2}}\end{pmatrix} = +\frac{1}{2}\hbar\cdot\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{i}{\sqrt{2}}\end{pmatrix}$$

$m_y = -\frac{1}{2}$ $$\frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\cdot\begin{pmatrix}\frac{1}{\sqrt{2}}\\ -\frac{i}{\sqrt{2}}\end{pmatrix} = -\frac{1}{2}\hbar\cdot\begin{pmatrix}\frac{1}{\sqrt{2}}\\ -\frac{-i}{\sqrt{2}}\end{pmatrix}$$

$\hat{J}_z \;\; for \;\; j=1$

$$\hat{J}_z = \hbar\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}$$

$$m_z = +1 \qquad \hbar\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}\cdot\begin{pmatrix}1\\ 0\\ 0\end{pmatrix} = +\hbar\cdot\begin{pmatrix}1\\ 0\\ 0\end{pmatrix}$$

$$m_z = 0 \qquad \hbar\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}\cdot\begin{pmatrix}0\\ 1\\ 0\end{pmatrix} = 0\cdot\begin{pmatrix}0\\ 1\\ 0\end{pmatrix}$$

$$m_z = -1 \qquad \hbar\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}\cdot\begin{pmatrix}0\\ 0\\ -1\end{pmatrix} = -\hbar\cdot\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}$$

$\hat{J}_x \;\; for \;\; j=1$

$$\hat{J}_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$$

$$m_x = +1 \qquad \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}\cdot\begin{pmatrix}\frac{1}{2}\\ \frac{1}{\sqrt{2}}\\ \frac{1}{2}\end{pmatrix} = +\hbar\cdot\begin{pmatrix}\frac{1}{2}\\ \frac{1}{\sqrt{2}}\\ \frac{1}{2}\end{pmatrix}$$

$$m_x = 0 \qquad \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}\cdot\begin{pmatrix}\frac{1}{\sqrt{2}}\\ 0\\ -\frac{1}{\sqrt{2}}\end{pmatrix} = 0\cdot\begin{pmatrix}\frac{1}{\sqrt{2}}\\ 0\\ -\frac{1}{\sqrt{2}}\end{pmatrix}$$

$$m_x = -1 \qquad \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}\cdot\begin{pmatrix}\frac{1}{2}\\ -\frac{1}{\sqrt{2}}\\ \frac{1}{2}\end{pmatrix} = -\hbar\cdot\begin{pmatrix}\frac{1}{2}\\ -\frac{1}{\sqrt{2}}\\ \frac{1}{2}\end{pmatrix}$$

$\hat{J}_y \;\; for \;\; j=1$

$$\hat{J}_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{pmatrix}$$

$$m_y = +1 \qquad \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{pmatrix}\cdot\begin{pmatrix}\frac{1}{2}\\ \frac{i}{\sqrt{2}}\\ -\frac{1}{2}\end{pmatrix} = +\hbar\cdot\begin{pmatrix}\frac{1}{2}\\ \frac{i}{\sqrt{2}}\\ -\frac{1}{2}\end{pmatrix}$$

$$m_y = 0 \qquad \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{pmatrix}\cdot\begin{pmatrix}\frac{1}{\sqrt{2}}\\ 0\\ \frac{1}{\sqrt{2}}\end{pmatrix} = 0\cdot\begin{pmatrix}\frac{1}{\sqrt{2}}\\ 0\\ \frac{1}{\sqrt{2}}\end{pmatrix}$$

$$m_y = -1 \qquad \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{pmatrix}\cdot\begin{pmatrix}\frac{1}{2}\\ -\frac{i}{\sqrt{2}}\\ -\frac{1}{2}\end{pmatrix} = -\hbar\cdot\begin{pmatrix}\frac{1}{2}\\ -\frac{i}{\sqrt{2}}\\ -\frac{1}{2}\end{pmatrix}$$