Quantum Physics 1

Questions and answer sets of mid-term exams

Version of September 2013

Contents

Midterm test for Kwantumfysica 1 - 2011-2012

Friday 30 September 2011, 14:00 - 15:00

<u>READ THIS FIRST</u>:

- Clearly write your name and study number on each answer sheet that you use.
- On the first answer sheet, write clearly the total number of answer sheets that you turn in.
- Note that this test has 5 questions, it continues on the backside of the paper!
- Start each question (number T1, T2,…) on a new side of an answer sheet.
- The test is open book within limits. You are allowed to use the book by Griffiths OR Liboff, and one A4 sheet with notes, but nothing more than this.
- If you get stuck on some part of a problem for a long time, it may be wise to skip it and try the next part of a problem first. The test is only 1 hour.

Useful formulas and constants:

Electron mass	m_e	$= 9.1 \cdot 10^{-31}$ kg
Electron charge	$-e$	$= -1.6 \cdot 10^{-19}$ C
Planck's constant	h	$= 6.626 \cdot 10^{-34}$ Js $= 4.136 \cdot 10^{-15}$ eVs
Planck's reduced constant	$\hbar$	$= 1.055 \cdot 10^{-34}$ Js $= 6.582 \cdot 10^{-16}$ eVs

Problem T1

For this problem, you must write up your answers in Dirac notation.

Consider a quantum system that contains a charged particle with mass m, that has a time-independent Hamiltonian

$$\hat{H} = \hat{T} + \hat{V} ,$$

where T a kinetic-energy term and V a potential-energy term. The two energy eigenstates of this system with the lowest energy are defined by

$$\begin{aligned} \hat{H}|\varphi_1\rangle &= E_1|\varphi_1\rangle \\ \hat{H}|\varphi_2\rangle &= E_2|\varphi_2\rangle \end{aligned} ,$$

where $E_1 < E_2$ the two energy eigenvalues, and $|\varphi_1\rangle$ and $|\varphi_2\rangle$ two orthogonal, normalized energy eigenvectors. Energy eigenstates with higher energy do not play a role. The observable $\hat{A}$ is associated with the electrical dipole moment A of this quantum system. For this system,

$$\langle\varphi_1|\hat{A}|\varphi_1\rangle = 0 \quad , \quad \langle\varphi_2|\hat{A}|\varphi_2\rangle = 0 \quad , \quad \langle\varphi_1|\hat{A}|\varphi_2\rangle = \langle\varphi_2|\hat{A}|\varphi_1\rangle = A_0 \ .$$

Note that the states $|\varphi_1\rangle$ and $|\varphi_2\rangle$ are energy eigenvectors, and that they are ***not*** eigenvectors of $\hat{A}$. At some time defined as $t = 0$, the state of the system is (with all c_n a complex-valued constant)

$$|\Psi_0\rangle = c_1\,|\varphi_1\rangle + c_2\,|\varphi_2\rangle = \frac{1}{\sqrt{2}}|\varphi_1\rangle - \frac{1}{\sqrt{2}}|\varphi_2\rangle \ .$$

Show that as a function of time $t > 0$, the expectation value for $\langle\hat{A}\rangle$ has oscillations at one frequency only. Determine this frequency and the oscillation amplitude (expressed in the constants that are mentioned above), for the case that the system is in $|\Psi_0\rangle$ at $t = 0$.

Z.O.Z.

Problem T2

An electron beam hits a wall with two parallel slits 1 and 2. A detector is located in some point P behind the wall with the two slits. The wavefunction Ψ_1 represents the electron state for electrons that arrive at the detection point P, for the case that the electron went through slit 1 and hits the detector. Similarly, Ψ_2 represents the state at P for the case that an electron went through slit 2. When only slit 1 is open, the detector P counts on average $N_1 = 100$ electrons per second. Due to an asymmetry in the system $|\Psi_2/\Psi_1| = a = 3.0$. What is the count rate of electrons in the detector at P when:
a) only slit 2 is open.
b) both slits are open and there is an interference maximum at point P.
c) both slits are open and there is an interference minimum at point P.

Problem T3

A particle of the mass m is moving in a one-dimensional potential $U(x) = kx^2/2$ (with $k > 0$, note that this system is a simple harmonic oscillator). Assume that the wave function of the ground state of this system is of the form of $\Psi(x) = A \exp(-\alpha x^2)$, where A and α are some constants. Use the time-independent Schrödinger equation to find the constant α and show that the energy E of this state is equal to

$$E = \tfrac{1}{2}\hbar\omega_0 \quad \text{with} \quad \omega_0 = \sqrt{\frac{k}{m}}$$

(ω_0 is here the classical angular frequency of oscillations).
Hint: fill $\Psi(x)$ in into the time-independent Schrödinger equation, simplify the equation, and note that the equation must hold for all x.

Problem T4

Assume that an operator $\hat{A}$ has the eigenfunctions φ_n and associated eigenvalues a_n. Prove that for any function f the next statement holds: $f(\hat{A})\,\varphi_n = f(a_n)\,\varphi_n$
Hint: use that f can be written as a power series

$$f(\hat{A}) = \sum_{l=0}^{\infty} c_l\left(\hat{A}^l\right)$$

and build your proof by checking the statement for each term.

Problem T5

A wide parallel beam of electrons (only motion in y-direction) with a velocity of $v_y = 600$ m/s is incident on a screen with a single narrow slit of with d. Behind this first screen there is a second screen for detection. The distance between the screen with the slit and the detection screen is $l = 1$ m. Using the uncertainty principle, make an *estimate* for the width of the slit d for which the width W of the image on the detection screen is the narrowest.
Hint: For electrons that just passed the screen, you can assume that for transverse motion in the beam the state of electrons is close to a state with minimum uncertainty.

Midterm test for Kwantumfysica 1 - 2012-2013

Friday 28 September 2012, 14:00 - 15:00

<u>READ THIS FIRST</u>:

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- If you get stuck on some part of a problem for a long time, it may be wise to skip it and try the next part of a problem first. The test is only 1 hour.
- When you turn in your problems, please **put your answer sheets in the order T1, T2, T3**…and **staple** them together.

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where $E_1 < E_2$ the two energy eigenvalues, and $|\varphi_1\rangle$ and $|\varphi_2\rangle$ two orthogonal, normalized energy eigenvectors. Energy eigenstates with higher energy do not play a role. The observable $\hat{A}$ is associated with the electrical dipole moment A of this quantum system. For this system,

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Note that the states $|\varphi_1\rangle$ and $|\varphi_2\rangle$ are energy eigenvectors, and that they are ***not*** eigenvectors of $\hat{A}$. At some time defined as $t = 0$, the state of the system is (with all c_n a complex-valued constant)

$$|\Psi_0\rangle = c_1\,|\varphi_1\rangle + c_2\,|\varphi_2\rangle = \sqrt{\tfrac{2}{3}}\,|\varphi_1\rangle \ + e^{i\varphi}\sqrt{\tfrac{1}{3}}\,|\varphi_2\rangle \quad .$$

Here φ (a real number) is the phase of the superposition at $t = 0$.

a) [2 points]
Show that as a function of time $t > 0$, the expectation value for $\langle\hat{A}\rangle$ has oscillations at one frequency only. Determine this frequency and the oscillation amplitude (expressed in the constants that are mentioned above), for the case that the system is in $|\Psi_0\rangle$ at $t = 0$.

Z.O.Z. for question b) of T1

b) [1 point]
At a time $t = 10$ ns one measures whether the system is in energy eigenstate $|\varphi_1\rangle$ or $|\varphi_2\rangle$. Calculate the probability for the measurement outcome that it is in state $|\varphi_2\rangle$.

c) [1 point]
For the case of question b), assume that the measurement result was that the system was in state $|\varphi_2\rangle$. Next, the measurement apparatus is switched off again at $t = 11$ ns. From that moment on the same Hamiltonian as before the measurement is valid for describing the system. Describe the state of the system for $t > 11$ ns. You can assume that it is an ideal measurement apparatus for doing quantum measurements.

Problem T2

Consider a quantum particle with mass m that can only move in the x-direction. It is at some moment in time in the state

$$\Psi(x) = \begin{cases} A\left(1-(x-a_2)^2\right) & \text{for } a_1 < x < a_3 \\ A\left(1-(x-b_2)^2\right) & \text{for } b_1 < x < b_3 \\ 0 & \text{for all other } x \end{cases}.$$

This is a normalized state. The constants are
$a_1 = -3$ nm, $a_2 = -2$ nm, $a_3 = -1$ nm,
$b_1 = +2$ nm, $b_2 = +3$ nm, $b_3 = +4$ nm,
$A = \sqrt{15/32}\ \text{nm}^{-5/2}$
(and the constant 1 in the equation has in fact the unit nm^2).

a) [1 point]
Determine the expectation value $\langle \hat{x} \rangle$ for this state (**hint:** first make a graph of $\Psi(x)$).

b) [1 point]
When one measures the position of the particle when it is in this state, what is the probability to get a measurement outcome between -0.6 nm and + 0.6 nm?

c) [1 point]
When one measures the position of the particle when it is in this state, what is the probability to get a measurement outcome between +3 nm and + 4 nm?

Problem T3
Consider the following quantum system: a particle with mass m that can only move in the x-direction. It is not a free particle, it experiences a position-dependent potential that is constant in time. This is a system with one degree of freedom and with a stationary Hamiltonian.

a) [1 point]
Assume that the particle moves in a potential $V(x) = B_0 \cos(3x) + K_0 x^2$. Write down the time-independent Schrödinger equation for this case, using a representation where all states and operators are expressed as functions of x. That is, you must write it out in a form that shows each term of the equation, and work out each term as a function of x in as much detail as possible with the information that is given.

b) [2 points]
Consider once more a quantum particle of this type, but now it moves in a different potential $V(x)$, that is also constant in time. Derive for this system the time-independent Schrödinger equation from the time-dependent Schrödinger equation. Use the x-representation.

Kwantum fysica 1 Mid term Exam
30 Sept 2011 ANSWERS

T1) $\langle \hat{A} \rangle (t) = \langle \Psi(t) | \hat{A} | \Psi(t) \rangle$, with $\begin{cases} |\Psi(t)\rangle = \hat{U}|\Psi_0\rangle = e^{-\frac{i}{\hbar}\hat{H}t}|\Psi_0\rangle \\ \langle \Psi(t)| = \langle \Psi_0 | \hat{U}^{+} = \langle \Psi_0 | e^{+\frac{i}{\hbar}\hat{H}t} \end{cases}$

$\Rightarrow \langle \hat{A} \rangle (t) = \langle \Psi_0 | \hat{U}^{+} \hat{A} \hat{U} | \Psi_0 \rangle$, use $\omega_1 = \frac{E_1}{\hbar}$, $\omega_2 = \frac{E_2}{\hbar}$

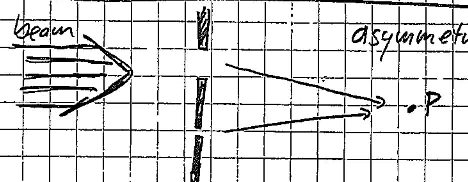
$\Rightarrow \langle \hat{A} \rangle (t) = \left(c_1^* e^{+i\omega_1 t} \langle \varphi_1 | + c_2^* e^{+i\omega_2 t} \langle \varphi_2 | \right) \hat{A} \left(c_1 e^{-i\omega_1 t} | \varphi_1 \rangle + c_2 e^{-i\omega_2 t} | \varphi_2 \rangle \right)$

$= |c_1|^2 \langle \varphi_1 | \hat{A} | \varphi_1 \rangle + |c_2|^2 \langle \varphi_2 | \hat{A} | \varphi_2 \rangle + c_1^* c_2 e^{-i(\omega_2 - \omega_1)t} \langle \varphi_1 | \hat{A} | \varphi_2 \rangle + c_2^* c_1 e^{+i(\omega_2-\omega_1)t} \langle \varphi_2 | \hat{A} | \varphi_1 \rangle$

$= 0 + 0 + (-1) \cdot \cos((\omega_2 - \omega_1)t) A_0$

$\Rightarrow$ Amplitude is A_0, only frequency is $f = \frac{\omega_2 - \omega_1}{2\pi} = \frac{E_2 - E_1}{2\pi \hbar}$

T2)



asymmetric double slit quantum interference experiment

a) $N_2 \propto |\Psi_2|^2 \Rightarrow N_2 = a^2 \cdot N_1 \Rightarrow$ 900 electrons/sec

b) $N_+ \propto |\Psi_1 + \Psi_2|^2 \Rightarrow N_+ = (a+1)^2 \cdot N_1 \Rightarrow$ 1600 electrons/sec

c) $N_- \propto |\Psi_1 - \Psi_2|^2 \Rightarrow N_1 = (a-1)^2 N_1 \Rightarrow$ 400 electrons/sec

T3) Time-indep. Schröd. Eq. $\left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{k}{2} x^2 \right) \Psi(x) = E \Psi(x)$

$\Psi = A e^{-\alpha x^2}$

$\frac{\partial}{\partial x} \Psi = -2\alpha A x e^{-\alpha x^2} \Rightarrow \frac{\partial^2}{\partial x^2} \Psi = -2\alpha A e^{-\alpha x^2} + 4\alpha^2 A x^2 e^{-\alpha x^2}$

Fill in into Schröd. Eq:

$-\frac{\hbar^2}{2m} (4\alpha^2 x^2 - 2\alpha)(Ae^{-\alpha x^2}) + \frac{k}{2} x^2 (Ae^{-\alpha x^2}) = E(Ae^{-\alpha x^2}) \Rightarrow$

$\left(-\frac{\hbar^2}{2m} 4\alpha^2 + \frac{k}{2} \right) x^2 + \frac{\hbar^2}{2m} 2\alpha - E = 0$

Z.O.Z.

Note that this must hold for all $x \Rightarrow$ ②

$\alpha^2 = \frac{mk}{4\hbar^2} \Rightarrow \alpha = \frac{\sqrt{mk}}{2\hbar}$

$E = \frac{2\alpha \hbar^2}{2m} \Rightarrow$ fill in $\alpha \Rightarrow E = \frac{\hbar}{2} \sqrt{\frac{k}{m}}$

T4) Given $\hat{A} \varphi_n = a_n \varphi_n$

As a power series, $f(\hat{A})$ can be written as

$f(\hat{A}) = \sum_{\ell=0}^{\infty} c_\ell (\hat{A}^\ell) \Rightarrow f(\hat{A}) \varphi_n = \sum_{\ell=0}^{\infty} c_\ell (\hat{A}^\ell \varphi_n)$

Use $\hat{A} \varphi_n = a_n \varphi_n$

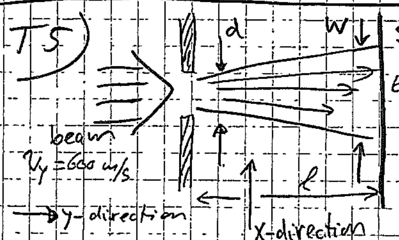
$\hat{A}^2 \varphi_n = \hat{A}(\hat{A}\varphi_n) = \hat{A}(a_n \varphi_n) = a_n \hat{A} \varphi_n = a_n^2 \varphi_n$

and so on for higher orders $\Rightarrow$

$(\hat{A}_n^\ell) \varphi_n = a_n^\ell \varphi_n \Rightarrow$

$f(\hat{A}) \varphi_n = \sum_{\ell=0}^{\infty} c_\ell (\hat{A}^\ell \varphi_n) = \left(\sum_{\ell=0}^{\infty} c_\ell a_n^\ell \right) \varphi_n = f(a_n) \varphi_n$

T5)



Right after the screen with the slit $\Delta x \simeq d \Rightarrow$

$\Delta p_x \approx \frac{\hbar}{2\Delta x} = \frac{\hbar}{2d}$

While flying to the detection screen, the beam will get wider from d to a width $W = d + \Delta W$.

$\Delta W = \Delta v_x \, t = \frac{\Delta p_x}{m} t$, where t the time of flight between the two screens. $\Rightarrow t = \frac{\ell}{v_y} \Rightarrow$

$W = d + \Delta W = d + \frac{\hbar}{2md} \cdot \frac{\ell}{v_y} \Rightarrow W$ has a minimum for a certain d. To find this d solve $\frac{dW}{dd} = 0 \Rightarrow$

$1 - \frac{1}{d^2} \frac{\hbar \ell}{2m v_y} = 0 \Rightarrow d \approx \sqrt{\frac{\hbar \ell}{2m v_y}}$

Answers to midterm exam KWANTUMFYSICA 1 28-9-2012

T1

a) $$\langle\hat{A}\rangle(t) = \langle\psi_0| e^{+\frac{i}{\hbar}\hat{H}t}\hat{A}\, e^{-\frac{i}{\hbar}\hat{H}t}|\psi_0\rangle =$$

$$\left(\sqrt{\tfrac{2}{3}}\, e^{+\frac{i}{\hbar}E_1 t}\langle\varphi_1| + e^{-i\varphi}\sqrt{\tfrac{1}{3}}\, e^{+\frac{i}{\hbar}E_2 t}\langle\varphi_2|\right)\hat{A}\left(\sqrt{\tfrac{2}{3}}\, e^{-\frac{i}{\hbar}E_1 t}|\varphi_1\rangle + e^{+i\varphi}\sqrt{\tfrac{1}{3}}\, e^{-\frac{i}{\hbar}E_2 t}|\varphi_2\rangle\right)$$

$$= \tfrac{2}{3}\langle\varphi_1|\hat{A}|\varphi_1\rangle + \tfrac{1}{3}\langle\varphi_2|\hat{A}|\varphi_2\rangle + \tfrac{\sqrt{2}}{3}e^{-i\left(\left(\frac{E_2-E_1}{\hbar}\right)t-\varphi\right)}\langle\varphi_1|\hat{A}|\varphi_2\rangle$$

$$+ \tfrac{\sqrt{2}}{3}e^{+i\left(\left(\frac{E_2-E_1}{\hbar}\right)t-\varphi\right)}\langle\varphi_2|\hat{A}|\varphi_1\rangle$$

$$= \frac{\sqrt{2}}{3}\left(e^{+i\left(\left(\frac{E_2-E_1}{\hbar}\right)t-\varphi\right)} + e^{-i\left(\left(\frac{E_2-E_1}{\hbar}\right)t-\varphi\right)}\right)A_0$$

$$= \frac{2\sqrt{2}}{3}A_0\cos\left(\frac{E_2-E_1}{\hbar}\cdot t - \varphi\right) \Rightarrow$$

It only oscillates at angular frequency $\frac{E_2-E_1}{\hbar}$ with amplitude $\frac{2\sqrt{2}}{3}A_0$.

b) $$P_2 = |\langle\Psi(t)|\varphi_2\rangle|^2 = \left|\left(\sqrt{\tfrac{2}{3}}\langle\varphi_1|e^{+\frac{i}{\hbar}E_1 t} + e^{-i\varphi}\sqrt{\tfrac{1}{3}}\langle\varphi_2|e^{+\frac{i}{\hbar}E_2 t}\right)|\varphi_2\rangle\right|^2$$

use $\langle\varphi_1|\varphi_2\rangle = 0$, $\langle\varphi_2|\varphi_2\rangle = 1$:

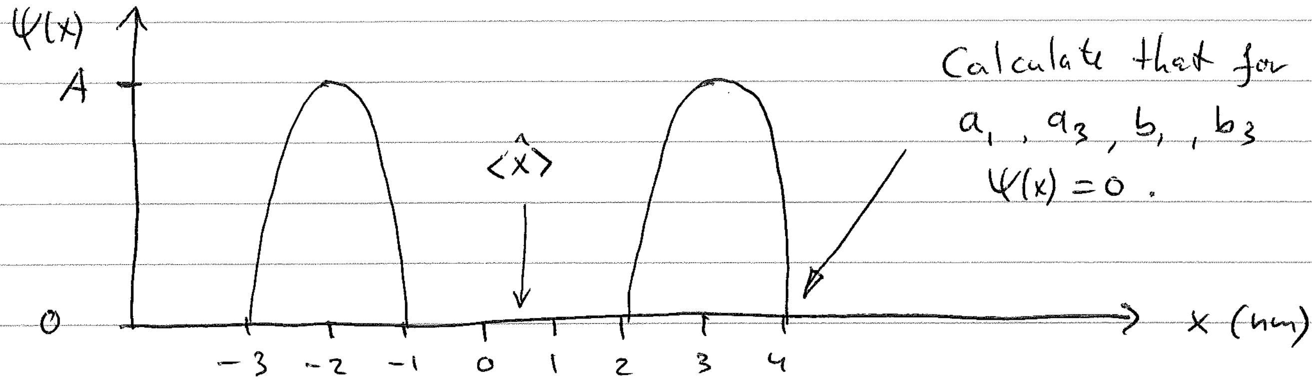
$$= \left|\sqrt{\tfrac{1}{3}}\, e^{i\left(\frac{E_2}{\hbar}t-\varphi\right)}\right|^2 = \frac{1}{3}$$

c) After the measurement the system is in state $|\varphi_2\rangle$

$$|\Psi(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}|\varphi_2\rangle = e^{-\frac{i}{\hbar}E_2(t-t_0)}|\varphi_2\rangle \quad \text{with } t_0 = 11\text{ ns}$$

This is effectively $|\Psi(t)\rangle = |\varphi_2\rangle$ since the prefactor is only a global phase factor.

T2 Make first a sketch of the wave function



a) $\langle\hat{x}\rangle = \int_{-\infty}^{\infty} \psi(x)^* x \psi(x)\, dx = \frac{1}{2}$ nm (Read from plot)

(or calculate it as

$$\langle\hat{x}\rangle = A^2\int_{a_1}^{a_3} \left(1-(x-a_2)^2\right)^2 x\, dx + A^2\int_{b_1}^{b_3} \left(1-(x-b_2)^2\right)^2 x\, dx$$)

b) Read from the plot that $|\psi(x)|^2$ is zero for $-0.6\,\text{nm} < x < 0.6\,\text{nm}$ ⇒ This probability is zero

(OR calculate it as $P = \int_{-0.6\,\text{nm}}^{+0.6\,\text{nm}} |\psi(x)|^2 dx = 0$)

c) Read from the plot that $3\,\text{nm} < x < 4\,\text{nm}$ concerns $\frac{1}{4}$ of the probability in the plot ⇒ the probability is $\frac{1}{4}$.

(OR calculate it as $P = \int_{3\,\text{nm}}^{4\,\text{nm}} |\psi(x)|^2 dx$

$$= \int_{b_2}^{b_3} A^2\left(1-(x-b_2)^2\right)^2 dx = \frac{1}{4}$$)

a) $\hat{H}\,\varphi(x) = E_i\,\varphi(x) \Rightarrow$

$$-\frac{\hbar^2}{2m}\frac{d^2\varphi(x)}{dx^2} + \left(B_0\cos(3x) + k_0x^2\right)\varphi(x) = E_i\,\varphi(x)$$

b) Use here the x-representation, for case that the single degree of freedom is the position in x direction of particle with mass m. (but you can work it out in a similar way for any other single degree of freedom)

Time-dependent Schrödinger equation: $i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \hat{H}\Psi(x,t)$ (1)

with for this case $\hat{H} = V(x) - \frac{\hbar^2}{2m}\frac{d^2}{dx^2}$.

Investigate whether there are solutions of the type $\Psi(x,t) = \varphi(x)\,\chi(t)$

Filling this in into Eq.(1), and deviding by $\varphi(x)\,\chi(t)$ gives

$$\frac{i\hbar}{\chi(t)}\frac{d\chi(t)}{dt} = \frac{1}{\varphi(x)}\left(-\frac{\hbar^2}{2m}\frac{d^2\varphi(x)}{dx^2}\right) + V(x)$$

This equality can only hold (left function of t only, right function of x only) if left and right are equal to a constant, which will be denoted as E_i. This gives two equations

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right)\varphi(x) = \hat{H}\varphi(x) = E_i\,\varphi(x)$$ → time-independent Schrödinger Eq.

$$i\hbar\frac{d\chi(t)}{\chi(t)} = E_i\,dt \Rightarrow i\hbar\ln(\chi(t)) = E_i\cdot t + C \Rightarrow \chi(t) = e^{-\frac{i}{\hbar}(E_i t + C)}$$ → time evolution of states with fixed E_i.