Problem set with Chapter 5 for werkcollege Kwantumfysica 1

Study year 2010-2011, see also werkcollege schedule with planning per group

Homework problems for this week:
5.2, 5.8, 5.12, 5.13, 5.19, 5.30, 5.33, 5.42, 5.46, 5.48, 5.50

Werkcollege problems for 2nd and 3rd werkcollege hour today:
C5.1, C5.2, C5.3, C5.4 and from the book 5.22, 5.28, 5.29, 5.37, 5.47, 5.52

Problem C5.1

This problem (about a particle in a box) is meant to clarify how the same *state* of quantum system can be *represented* in many different ways. The four representations that will be used here are:
- describing a state using Dirac notation
- a wavefunction that is a function of position x
- a wavefunction that is a function of wave number k (or, equivalently, momentum $p_x=\hbar k$)
- a superposition of energy eigenstates

On the way, you will practice with Fourier transforms and decomposition of a state into eigenvectors.

a) The particle in the box is modeled as a particle in an infinitely deep potential well with $V=0$ for $|x|<a/2$, and $V=\infty$ elsewhere. The particle is brought into the box with a mechanism that results in a wavefunction for the particle that is evenly distributed in the well, $\Psi(x)=1/\sqrt{a}$ for $|x|<a/2$ and zero elsewhere. Represent this state in the k-representation (hint: you need to Fourier transform the state).

b) Alternatively, this state can for example be represented as a superposition of energy eigenstates of the system in Dirac notation, $|\Psi\rangle = \sum_n c_n |\varphi_n\rangle$. We will use this later in this problem. In this question we first we pay attention to the relation between the x-representation and the representation with Dirac notation. Prove the relation $\Psi(x)=\langle x|\Psi\rangle$ ($|x\rangle$ is the eigenvector with eigenvalue x for the position operator $\hat{x}$ on p. 74).

c) Equation (6.100) in the book gives the energy eigenstates for this system. Write down the eigenfunctions for odd n in the k-representation.

d) Sketch the wavefunction of the particle (see **a)**), as well as the energy eigenstates $|\varphi_1\rangle$ and $|\varphi_9\rangle$ in the k-representation (see book p. 122 for example of graphs of sinc functions). Explain the differences between the graphs.

e) For continuing on **b)**, you need to determine the coefficients c_n for odd n. Proof the relation $c_n=\langle\varphi_n|\Psi\rangle$ in Dirac notation.

f) Evaluate the inner product $c_n=\langle\varphi_n|\Psi\rangle$ for odd n in the x-representation.

g) Write down the inner product $c_n=\langle\varphi_n|\Psi\rangle$ for odd n in the k-representation, but only solve the integral if you feel like doing so.

h) Without doing the calculation, can you say what the value is of $c_n=\langle\varphi_n|\Psi\rangle$ for even n.

NOTE: Fourier transform relation between x- and k-representation of a state:

$$\Psi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\overline{\Psi}(k)e^{ikx}\,dk$$

$$\overline{\Psi}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Psi(x)e^{-ikx}\,dx$$

Problem C5.2
The wavefunction of a system can be expanded, depending on the problem, either over a discrete basis set, or over a continuum of states. In the first case (valid for example for the infinite well potential) we write

$$\Psi(x) = \sum_n b_n \varphi_n(x),$$

where the φ_n are, for instance, energy eigenstates, while in the second case the proper expression is an integral:

$$\Psi(x) = \int_{-\infty}^{\infty} b(k)\varphi_k \, dk,$$

where for example the φ_k are momentum eigenfunctions for a free particle.

a) What are the dimensions of $\psi(x)$ and φ_k? (Hint: use the normalization condition for the wave function)

b) What are the dimensions of $(b_n)^2$ and $|b(k)|^2$, respectively?

c) Using equations (5.25) and (5.26) in the book, show (and have a look at Appendix C in the book for some help on the integrals) that

$$\langle \Psi | \Psi \rangle = 1 \rightarrow \int_{-\infty}^{+\infty} |b(k)|^2 dk = 1$$

Problem C5.3
Measurement of the position of a particle in a one-dimensional box with walls at x = 0 and x = a finds the value x = a/2.

a) What is the wavefunction of the system immediately after the measurement?

b) By expanding this wavefunction of **a)** over the eigenstates of the energy operator Ĥ (page 93 of the book), show that in a subsequent measurement of energy, it is equally probable to find the particle in any odd-energy eigenstate (an eigenstate φ_n is odd if n is odd and even if n is even), while the probability of finding the particle in any given even eigenstate is zero.

c) Again applied to the same state of **a)**, show that the total probability for a measurement of energy is not normalized. Does this result surprise you?

Problem C5.4
Consider an electron, that behaves as a one-dimensional quantum particle.

a) At some time t_0 the electron is in the state $\Psi(x) = Ae^{\frac{-|x|}{a}}$, where A real and positive. For which value of A is this state normalized?

b) Derive an expression that describes this state as a superposition of plane waves with wavenumber k.

c) *Roughly estimate* Δx and Δp_x for the state of question **a)**, and check whether it violates the Heisenberg uncertainty relation.

d) One wants to measures the velocity of this particle at time t_0. Calculate the probability for getting a result between 40 km/s and 50 km/s. Calculate a numerical result, use a = 1 nm. You may need to use this solution to the following integral,

$$\int \left(\frac{1}{1+b^2y^2} \right)^2 dy = \frac{1}{2}\left(\frac{y}{1+b^2y^2} + \frac{\arctan(by)}{b} \right).$$