Final exam for Kwantumfysica 1 - 2005-2006

Monday 26 June 2006, 9:00 - 12:00

<u>READ THIS FIRST</u>:

- Clearly write your name and study number on each answer sheet that you use.
- On the first answer sheet, write clearly the total number of answer sheets that you turn in.
- Note that this exam has 4 questions, it continues on the backside of the papers!
- Start each question (number 1, 2, 3, 4) on a new answer sheet.
- The exam is open book. You are also allowed to use formula sheets etc.
- If it says "make a rough estimate", there is no need to make a detailed calculation, and making a simple estimate is good enough. If it says "calculate" or "derive", you are supposed to present a full analytical calculation.
- If you get stuck on some part of a problem for a long time, it may be wise to skip it and try the next part of a problem first.

Useful formulas and constants:

Electron mass	m_e	$= 9.1 \cdot 10^{-31}$ kg
Electron charge	$-e$	$= -1.6 \cdot 10^{-19}$ C
Planck's constant	h	$= 6.626 \cdot 10^{-34}$ Js $= 4.136 \cdot 10^{-15}$ eVs
Planck's reduced constant	$\hbar$	$= 1.055 \cdot 10^{-34}$ Js $= 6.582 \cdot 10^{-16}$ eVs

Fourier relation between x-representation and k-representation of a state

$$\Psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{\Psi}(k) e^{ikx}\, dk$$

$$\overline{\Psi}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x) e^{-ikx}\, dx$$

Z.O.Z.

Problem 1

A box of 1 mm by 1 mm by 1 mm contains two quantum particles (labeled i = 1, 2) that do not interact. They are described by the position operators $\hat{x}_1, \hat{y}_1$ and $\hat{z}_1$, and $\hat{x}_2, \hat{y}_2$ and $\hat{z}_2$, and momentum operators $\hat{p}_{x1}, \hat{p}_{y1}$ and $\hat{p}_{z1}$, and $\hat{p}_{x2}, \hat{p}_{y2}$ and $\hat{p}_{z2}$.

a) What is the number of degrees of freedom of this system?

b) Discuss which of these 12 operators commute, and which do not commute. List the commutators of pairs that do not commute. Explain your answer by discussing which observables can be measured at the same time with high accuracy.

Problem 2

A free one-dimensional quantum particle with mass m, is propagating through space (position x, wavenumber k), and has a time-dependent wavefunction (time t) that is described as

$$\Psi(t) = \int_{k_1}^{k_2} dk\, A(k) \exp(i(kx - \omega t)).$$

Here $A(k)$ is a smooth envelope function that is non-zero in the range k_1 until k_2, with $0<k_1<k_2$.

a) Describe the physics of this state in words.

b) Discuss what the phase velocity is of the particle in this state.

c) How does ω depend on k? Explain your answer.

d) What is the group velocity of this particle?

Problem 3

An apparatus for an experiment shoots out electrons one by one, in one particular direction (the x-direction, the direction of the beam). The beam is aimed at a screen that contains a small slit, with a width of d = 100 nm in y-direction, centered at y_0 = 0 m. The electrons either hit the screen, or pass through the slit and are then a free particle. We consider here the electrons that passed the screen. These electrons have directly behind the screen a block-shaped wavefunction $\psi(y)$ that describes the y-position of the electron, with a width of 100 nm. One can assume that this wavefunction has a real and positive amplitude at that moment.

a) Sketch this wavefunction $\psi(y)$.What is at that moment the amplitude near y = 0 m?

b) Give the k_y-representation of this state (k_y is the wavenumber in y-direction).

c) Estimate what the uncertainty in y-momentum Δp_y is for electrons that just passed the screen.

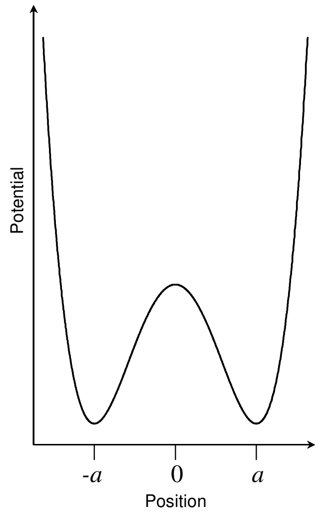
d) How does Δp_y depend on time, after electrons passed the screen?

e) Make a rough estimate of the width W of the wavefunction $\psi(y)$ of an electron, for the case that a time t_p has passed since the electron passed the screen. Describe how W depends on d and t_p.

f) The slit is moved sideways, and centered at y_1 = 1 mm. The amount of electrons passing the screen stays the same since the incident bundle is quite wide. One can still assume that the wavefunction $\psi(y)$ has immediately after the screen a block shape with real and positive amplitude. What is now the k_y-representation of this state, for electrons that pass the screen? Discuss the physical differences with your answer on question **b)**.

Problem 4

In a molecule, an electron is tightly bound to the other particles in the system. In one direction, it can be in either in one of two positions, because the electron experiences in this direction a one-dimensional potential as in the following sketch.



The barrier between the two wells is so high, that tunneling between the left and right well is negligible. In this situation, the system has two energy eigenstates with the same energy E_0. One of these states, denoted as $|\varphi_L\rangle$, corresponds to the particle being localized at $-a$ in the left well. The other energy eigenstate, denoted as $|\varphi_R\rangle$, corresponds to the particle being localized at $+a$ in the right well. All other energy eigenstates are so high in energy that they do not need to be considered. The system can therefore be described as a two-state system. Besides Dirac notation, it is convenient to use matrix and vector notation in the basis spanned by $|\varphi_L\rangle$ and $|\varphi_R\rangle$, which gives the following relations ($\hat{H}_0$ is the Hamiltonian)

$$\hat{H}_0 \leftrightarrow \begin{pmatrix} E_0 & 0 \\ 0 & E_0 \end{pmatrix}, \quad |\varphi_L\rangle \leftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\varphi_R\rangle \leftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

a) Use the matrix notation as defined here, to verify that the system has indeed two energy eigenvalues with energy E_0. Explain how you get to your answer.

b) Explain whether the energy eigenvalue E_0 is degenerate.

When the molecule is placed in an electric field of 1 V/mm, the only effect on the potential for the electron is that barrier between the two wells becomes lower. In that case tunneling between the two wells can no longer be neglected when describing the dynamics of the electron. Using the same matrix notation as before (also in the same basis), the Hamiltonian of the system is now (here T is a real and positive number)

$$\hat{H} \leftrightarrow \begin{pmatrix} E_0 & T \\ T & E_0 \end{pmatrix}.$$

c) Read question d) and e) before answering this question. Use the matrix notation to verify that the energy eigenstates of this new Hamiltonian $\hat{H}$ are $|\Psi_1\rangle = (|\varphi_L\rangle + |\varphi_R\rangle)/\sqrt{2}$ and $|\Psi_2\rangle = (|\varphi_L\rangle - |\varphi_R\rangle)/\sqrt{2}$.

d) What are now the energy eigenvalues of the system?

e) Proof that the new energy eigenstate $|\Psi_1\rangle$ is normalized.

f) At some moment in time defined as $t = 0$, the external electric field is off, and the electron is in the left well in the state $|\varphi_L\rangle$. At this moment, the external electric field is suddenly switched on. Describe (in words) how the position of the electron evolves for time $t > 0$. Explain your answer in as much detail as you can give.

g) At some later moment in time defined as $t = t_1$, the external electric field is still on, and the system has relaxed to the ground state. Describe (in words) how the position of the electron evolves for time $t > t_1$. Explain your answer.