Final exam for Kwantumfysica 1 - 2005-2006

Wednesday 26 April 2006, 14:00 - 17:00

<u>READ THIS FIRST</u>:

- Clearly write your name and study number on each answer sheet that you use.
- On the first answer sheet, write clearly the total number of answer sheets that you turn in.
- Note that this exam has 3 questions, it continues on the backside of the papers!
- Start each question (number 1, 2, 3) on a new answer sheet.
- The exam is open book. You are also allowed to use formula sheets etc.
- If it says "make a rough estimate", there is no need to make a detailed calculation, and making a simple estimate is good enough. If it says "calculate" or "derive", you are supposed to present a full analytical calculation.
- If you get stuck on some part of a problem for a long time, it may be wise to skip it and try the next part of a problem first.
- If you are ready with the exam, please fill in the **course-evaluation question sheet**. You can keep working on the exam until 17:00, and fill it in after shortly after 17:00 if you like.

Useful formulas and constants:

Electron mass	m_e	$= 9.1 \cdot 10^{-31}$ kg
Electron charge	$-e$	$= -1.6 \cdot 10^{-19}$ C
Planck's constant	h	$= 6.626 \cdot 10^{-34}$ Js $= 4.136 \cdot 10^{-15}$ eVs
Planck's reduced constant	$\hbar$	$= 1.055 \cdot 10^{-34}$ Js $= 6.582 \cdot 10^{-16}$ eVs

Fourier relation between x-representation and k-representation of a state

$$\Psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{\Psi}(k) e^{ikx} \, dk$$

$$\overline{\Psi}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x) e^{-ikx} \, dx$$

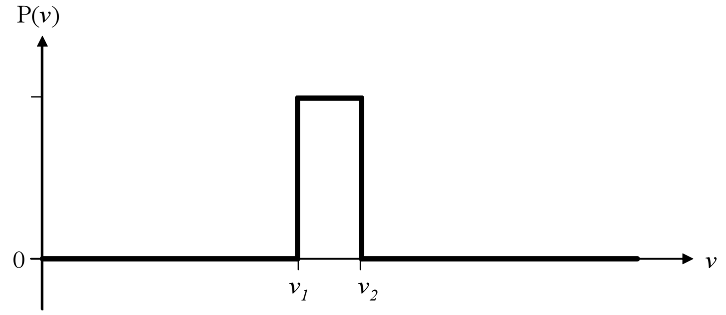
Z.O.Z.

Problem 1

An apparatus for an experiment shoots out electrons one by one, in one particular direction (the x-direction, the direction of the beam). After leaving the apparatus, the beam is passing an area where no significant forces act on the electron. Consequently, the electrons in the beam can be described as a free particle moving in one dimension.

a) Write down a Hamiltonian for one of the electrons in the beam. Explain your answer.

The research team aims at setting up the experiment in such a way that all the electrons leave the apparatus at the same speed, and that the quantum uncertainty in the speed of each electron is quite small. They aim at giving the electrons a velocity of 100 m/s. To check whether the setup works, they measure at some point in the beam (which they will define as $x = 0$) the velocity of a large number of electrons. They find a probability distribution $P(v)$ for the electrons' velocities v as in the figure below (uniform, with $v_1 = 99$ m/s and $v_2 = 101$ m/s).



b) They now remove this measurement apparatus from the beam, such that electrons passing $x = 0$ are not disturbed. Describe and sketch a normalized wavefunction $\overline{\Psi}(k)$ as a function of wavenumber k (in x-direction) for one of the electrons while it passes $x = 0$ (put in the sketch labels k_1 and k_2, related to v_1 and v_2). Use a wavefunction which agrees with the observed distribution $P(v)$, and show that it is normalized. Assume that the wavefunction can be chosen real and positive where it is not zero.

c) Calculate the wavefunction as a function of position x, that describes state of the electron which you already described as a function of k for answering question **b)**. Hint: write k_1 as k_c-Δk and k_2 as k_c+Δk (with $k_c = (k_1+k_2)/2$, and $\Delta k = (k_2+k_1)/2$).

d) Sketch and describe the probability density as a function of x, for the wavefunction you found in answer **c)**.

e) Estimate the uncertainty in the momentum and the position of the particle when it is near $x = 0$. Evaluate your answer.

f) They now put the measurement apparatus back into the beam. It measures the speed of the electrons without trapping the electron (that is, the electrons continue to fly along the beam). At some moment, an electron passes the measurement apparatus, and the velocity is measured with very high accuracy, with result v_m. Explain why one should now assume that the wavefunction as a function of k is very close to the form $\overline{\Psi}(k) = \delta(k - k_m)$. Calculate k_m is the result is 99.5 m/s.

g) For the case of **f)**, consider the moment that the measurement was just finished, and the electron is still near $x = 0$. For this situation, calculate the wavefunction as a function of x that describes the state of the electron.

h) Sketch the probability density as a function of x, for the wavefunction you found in answer **g)**.

i) Evaluate the validity of the description of the state of the system in answers **f)-h)**.

j) One team members suggests to keep the velocity measurement apparatus in the beam, as it reduces the quantum uncertainty in the velocity of the electrons. Discuss whether this indeed is useful for meeting the goals that have been summarized above the figure. Discuss it for individual electrons, and for the ensemble of electrons.

Problem 2

Note: Use Dirac notation for solving this problem.

Consider a system with a time-independent Hamiltonian $\hat{H}$, that has only two energy eigenstates. These have two different energies E_1 and E_2 with $E_2 > E_1$.

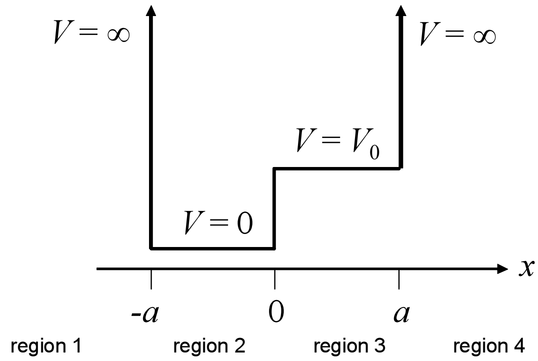
a) Write down both the time-dependent and the time-independent Schrödinger equation for this system in Dirac notation.

b) The time evolution of this system can be described using the time-evolution operator $\hat{U} = e^{\frac{-i\hat{H}t}{\hbar}}$. Derive this operator from the time-dependent Schrödinger equation.

c) This system has an electrical dipole moment that is described by the operator $\hat{D}$. For this system, this operator $\hat{D}$ commutes with $\hat{H}$. Assume that at some point in time $t = 0$, the system is in some arbitrary superposition of the two energy eigenstates. Show that this system will then never have any oscillations of $\langle \hat{D} \rangle$ in time. Use Dirac notation and the operator $\hat{U} = e^{\frac{-i\hat{H}t}{\hbar}}$.

Problem 3

In a molecule, an electron is tightly bound to the other particles in the system. In one direction, however, it is free to move a little bit from one atom to a neighboring atom. Along this direction, the electron experiences a one-dimensional potential *V(x)* as a function of position *x*. The potential *V(x)* can be approximated very well by the potential landscape as in the following sketch.



a) Give the Hamiltonian for this system, with the potential *V(x)* written out for each region along *x*.

b) It turns out that the energy for the ground state of this system $E_g > V_0$. Consequently, a general form for the part of the wavefunctions of the energy eigenstates in region 2 will be $\varphi_2(x) = Ae^{ik_2x} + Be^{-ik_2x}$, while for region 3 it will be $\varphi_3(x) = Ce^{ik_3x} + De^{-ik_3x}$. In regions 1 and 4 the wavefunctions will be zero. Explain why this can be assumed for regions 1, 2, 3 and 4 (see also question **c)**).

c) Give expressions for k_2 and k_3. Show how these expressions can be derived from the time-independent Schrödinger equation.

d) To find the energy eigenvalues and eigenfunctions of this system, on needs to write down equations that can be used to solve for *A*, *B*, *C* and *D*. Explain how one can define the set of equations needed to solve this problem, and give these equations (do not worry about normalization of the eigenstates yet).

e) Show that working out this problem of **d)** is equivalent to solving the following linear algebra problem: $\mathbf{M}\vec{v} = \vec{s}$, with

$$\mathbf{M} = \begin{pmatrix} 1 & 1 & -1 & -1 \\ ik_2 & -ik_2 & -ik_3 & ik_3 \\ e^{-ik_2a} & e^{ik_2a} & 0 & 0 \\ 0 & 0 & e^{ik_3a} & e^{-ik_3a} \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix}, \quad \vec{s} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

f) Use qualitative reasoning to find out what the shape is of the wavefunction for ground state and the first excited state as a function of *x*. Draw a sketch for these two wavefunctions, and explain your answer. Hint: consider this system as the result of two coupled particle-in-a-box systems, where the width of the tunnel barrier between the two boxes is reduced to zero, and the degeneracy of the two boxes is lifted. Also use the answer on **c)** if needed.